Exercise solutions: concepts from chapter 9

1) Data from rock mechanical testing of the Shiiya shale (Laboratory name, XT) a Japanese sedimentary rock, are reported in non-SI units and given in terms of confining pressures, $P_c$, and differential strengths in compression, $D_c$, by Hoshino et al. (1972).

a) Convert these data to SI units. Calculate the two unique principal stresses, $\sigma_1 = \sigma_2$ and $\sigma_3$, associated with each failure state, based on the values of confining pressure and differential strength. Show how the differential strength and triaxial compressive strength are related to the principal stresses.

The conversion factors are found as follows:

\[
1 \text{ kgf/cm}^2 \times 9.8 \times 10^4 \text{ cm}^2 = 9.8 \text{ N/m}^2
\]

\[
1 \text{ kgf/cm}^2 = 9.8 \times 10^{-2} \text{ MPa}
\]

Combining these two factors we find:

\[
1 \text{ kgf/cm}^2 \times 9.8 \times 10^{-2} \text{ MPa} = 9.8 \times 10^{-2} \text{ MPa}
\]

Therefore, one multiplies the confining pressures, $P_c$, and differential strengths, $D_c$, as reported by Hoshino et al. (1972) by the numerical factor $9.8 \times 10^{-2}$ to convert to the SI unit MPa. To calculate the two unique principal stresses at failure use (9.20) and (9.22):

\[
\sigma_1 = -P_c, \; \sigma_3 = -\left(D_c + P_c\right)
\]

Note that $\sigma_1$ acts in a radial direction on the sample and $\sigma_3$ acts in the axial direction. From (3) the differential strength in compression is related to the principal stresses as:

\[
D_c = -\left(\sigma_3 - \sigma_1\right), \; \text{at failure}
\]

The triaxial compressive strength is the magnitude of the axial stress at failure:

\[
C_t = -\sigma_3, \; \text{at failure}
\]

The relevant quantities for the triaxial compression tests to failure are given in the SI unit MPa in the following table. Note the differences between the differential strengths and triaxial compressive strengths for all cases except that with negligible confining pressure.

<table>
<thead>
<tr>
<th>$P_c$</th>
<th>$D_c$</th>
<th>$\sigma_1$</th>
<th>$\sigma_3 = -C_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.098</td>
<td>150</td>
<td>-0.098</td>
<td>-150</td>
</tr>
<tr>
<td>49</td>
<td>292</td>
<td>-49</td>
<td>-341</td>
</tr>
<tr>
<td>98</td>
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</tr>
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<td>147</td>
<td>409</td>
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<td>-556</td>
</tr>
<tr>
<td>196</td>
<td>461</td>
<td>-196</td>
<td>-657</td>
</tr>
<tr>
<td>245</td>
<td>564</td>
<td>-245</td>
<td>-809</td>
</tr>
</tbody>
</table>
b) Plot the strength data for the Shiiya shale in principal stress space. Determine the best fitting straight line to define this part of the failure surface and write down the equation for this line. Use the linear fit to the data and the Coulomb failure criterion to determine the uniaxial compressive strength, $C_u$, and coefficient of internal friction, $\mu_i$.

The MATLAB m-script `fig_09_sol_1.m` (see above) was used to plot the data and the MATLAB Basic Fitting tool was used to determine this segment of the failure surface.

![Figure 1. Plot of triaxial test data for Shiiya shale (blue circles) in principal stress space with the best fitting linear relationship (red line). Note the scales are different for the ordinate and abscissa.](image)

In terms of the principal stresses this segment of the failure surface is:

$$\sigma_3 = -190 \text{ MPa} + 2.52 \sigma_1$$

(6)

The linear relation based upon Coulomb failure theory is (9.50):

$$\sigma_3 = -C_u + \left[ \left(1 + \mu_i^2 \right)^{1/2} + \mu_i \right]^2 \sigma_1$$

(7)

Here $C_u$ is the uniaxial compressive strength (the negative of the intercept on the $\sigma_3$ axis) and $\mu_i$ is the coefficient of internal friction (related to the slope of the best fit line). Note that the best fit line estimates a uniaxial compressive strength of 190 MPa whereas the measured laboratory value is 150 MPa. Writing the slope of the best fit line as $m$, the coefficient of internal friction is found from (7) as:
\[ \left(1 + \mu_i^2\right)^{1/2} + \mu_i = \sqrt{m}, \text{ so } 1 + \mu_i^2 = \left(\sqrt{m} - \mu_i\right)^2 = m - 2\mu_i\sqrt{m} + \mu_i^2 \]  
\[ 2\mu_i\sqrt{m} = m - 1, \text{ so } \mu_i = \frac{m - 1}{2\sqrt{m}} \]

Values of the coefficient of internal friction are compiled in Table 9.7 for other rocks.

c) Calculate the differential strength and the triaxial compressive strength for the Shiiya shale for conditions of confining pressure that would be found at 5 km depth. Suggest reasons why this value is different from the uniaxial compressive strength based upon your understanding of the microscopic mechanisms that operate to limit the load-carrying capacity of rock subject to confining pressure.

At 5 km depth the confining pressure is approximated as:
\[ P_c \approx (25 \text{ MPa/km}) (5 \text{ km}) = 125 \text{ MPa} = -\sigma_i \]  
\[ \sigma_1 = -190 \text{ MPa} + (2.52)(-125 \text{ MPa}) = -505 \text{ MPa} \]

This is consistent with the graphical result (Figure 1). The differential strength is calculated using (4):
\[ D_c = \sigma_i - \sigma_3 = -125 \text{ MPa} + 505 \text{ MPa} = 380 \text{ MPa} \]

This differential strength is about twice the uniaxial compressive strength. Greater confining pressure pushes crack walls and grain boundaries together providing greater frictional resistance to sliding. Also local tensile stress perturbations must overcome the greater confining pressure before cracks will open and propagate. Both of these micromechanical mechanisms play important roles in the development of shear fractures which therefore are inhibited by the greater confining pressure.

d) Construct a line representing the lithostatic stress state on the graph from part b) of principal stress space with the failure surface for the Shiiya shale. Draw a path representing the stress state changes during the laboratory experiment. Draw a path that the stress state might have followed during burial and deformation to reach the differential strength at 5 km depth. Discuss possible effects of stress path on strength.

The graph of principal stress space is shown in Figure 2. The lithostatic stress state is represented by a straight line where \( \sigma_1 = \sigma_3 \). In most triaxial tests the confining pressure and axial compression are increased simultaneously so the stress path follows the lithostatic line until the confining pressure selected for the strength test is achieved. Then the axial compression is increased while the confining pressure is held constant until failure occurs. Many stress paths are possible in nature. The one drawn here closely approximates the lithostatic stress path initially, but then diverges such that the greater compression increases more rapidly. Both principal stresses achieve greater compressive values than the lab test before there is a reversal in the lesser compression as the greater compression continues to increase to failure. Because one would expect the strength to be effected by any micromechanical damage to the specimen during loading, the strength
may be a function of the stress path. It seems unlikely that natural stress paths would be bilinear with a sharp kink as is the case for the lab experiments.

Figure 2. Principal stress space with failure surface (red line segment) for Shiiya shale. Also shown are lithostatic stress states, the stress path for triaxial experiments, and a possible “natural” stress path.

e) Consider the strength of Shiiya shale in triaxial experiments as defined by the linear segment of the failure surface from part b) of this exercise. Plot six points on the failure surface in principal stress space that are the predicted triaxial strengths for confining pressures representative of 0, 1, 2, 3, 4, and 5 km depths with no pore fluid pressure. Using Terzaghi’s principal subtract pore pressures of 0, 10, 20, 30, 40, and 50 MPa from the respective principal stresses to plot six points on a new failure surface for Shiiya shale with pore fluid pressure on the same graph. Note that these pore pressures are what you would expect for hydrostatic columns of water to depths of 0, 1, 2, 3, 4, and 5 km. Based on the new confining pressures, however, calculate the new representative depths for these tests and the corresponding depths to the top of the water table.

According to Terzaghi’s concept the effective principal stresses are defined as:

\[ \sigma_i' = \sigma_i + P_p, \quad \sigma_3' = \sigma_3 + P_p \quad (13) \]

Here \( P_p \) is the pore pressure. The proposition is that the triaxial compressive strength of porous saturated rock is reduced by the magnitude of the pore pressure. Thus, all data on the stress states at failure for a particular rock subject to a variety of confining and pore pressures should reduce to the same failure surface when plotted in effective stress space. Given the segment of the failure surface in two-dimensional stress space for unsaturated Shiiya shale as defined in (6) one must subtract the pore pressure from both principal stresses at failure to calculate a position on the segment of the failure surface for that pore pressure:

\[ \sigma_i(P_p) = \sigma_i(P_p = 0) - P_p, \quad \sigma_3(P_p) = \sigma_3(P_p = 0) - P_p \quad (14) \]
For a hydrostatic pore pressure the gradient would be approximately 10 MPa/km, whereas for a lithostatic pore pressure the gradient would be approximately 25 MPa/km.

The MATLAB m-script fig_09_sol_3.m calculates the principal stresses on the failure surfaces for hydrostatic and lithostatic pore pressures and plots them.

```matlab
% fig_09_sol_3.m
% Triaxial strength of Shiiya shale without and with pore pressure
clear all, clf reset % clear functions and figures
pcg = 25; ppg = 10; % rock and water pressure gradients
D = 0:1:5; % depth intervals
PC = pcg*D; PP = ppg*D;
S1 = -PC; S3 = -190 + 2.52*S1; % principal stress at failure
S1H = S1 - PP; S3H = S3 - PP; % hydrostatic pore pressure
plot(S1,S3,'ro',S1H,S3H,'bx');
legend('unsaturated','hydrostatic');
xlabel('S1 (MPa)'), ylabel('S3 (MPa)');
title('Shiiya shale without and with pore pressure');
axis([-200 0 -600 0]);
```

Figure 3. Principal stress space with failure surfaces for Shiiya shale subject to zero and hydrostatic pore pressures. Note the different scales used for the two principal stresses.

The representative depths for points on the new failure surface are 0, 1.4, 2.8, 4.2, 5.6, and 7.0 km. Corresponding depths to the water table are 0, 0.4, 0.8, 1.2, 1.6, and 2.0 km. At a particular confining pressure, $P_c = -\sigma_i$, Figure 3 shows that the triaxial compressive strength, $C_i = -\sigma_i$, decreases as pore pressure increases. If the two line segments were re-plotted in effective stress space all of the symbols would collapse onto one line segment. To do this one would add the appropriate pore pressure to both principal stresses following (13) and this reverses the calculation made using (14).
2) Data from rock mechanical testing of the Maze sandstone (Laboratory name, XC) a Japanese sedimentary rock, are reported in non-SI units and given in terms of confining pressures, $P_c$, and differential strengths in compression, $D_c$, by Hoshino et al. (1972).

a) Convert these data to SI units. Calculate the two principal stresses, $\sigma_1$ and $\sigma_3$, associated with each failure state based on the values of confining pressure and differential strength and plot these data in principal stress space.

Procedures described under question 1 part a) were used to convert the units to SI, calculate the principal stresses, and plot these data (Fig. 4). The relevant quantities for the triaxial compression tests to failure are given in the SI unit MPa in the following table.

<table>
<thead>
<tr>
<th>$P_c$</th>
<th>$D_c$</th>
<th>$\sigma_1$</th>
<th>$\sigma_3$ = $-C_t$</th>
</tr>
</thead>
<tbody>
<tr>
<td>0.0980</td>
<td>113</td>
<td>-0.0980</td>
<td>-113</td>
</tr>
<tr>
<td>49</td>
<td>292</td>
<td>-49</td>
<td>-341</td>
</tr>
<tr>
<td>98</td>
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<td>-98</td>
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<td>147</td>
<td>422</td>
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<td>-569</td>
</tr>
<tr>
<td>196</td>
<td>507</td>
<td>-196</td>
<td>-703</td>
</tr>
</tbody>
</table>

Figure 4. Plot of triaxial test data for Maze sandstone (green circles) in principal stress space with the best fitting linear relationship (red line). Note the scales are different for the two principal stresses.

b) Use the best linear fit to these data as plotted in part a) and the Coulomb failure criterion to determine the uniaxial compressive strength, $C_u$, and coefficient of internal friction, $\mu_i$. Use these parameters to compute the inherent shear strength of the Maze sandstone. Write the Coulomb criterion for the Maze sandstone using these parameters.
In terms of the principal stresses the linear segment of the failure surface shown in Figure 4 is defined as:

$$\sigma_3 = -162 \text{ MPa} + 2.88 \sigma_1$$  \hspace{1cm} (15)

The uniaxial compressive strength is the negative of the intercept on the $\sigma_3$ axis and the coefficient of internal friction is related to the slope of the best fit line. The best fit line estimates a uniaxial compressive strength:

$$C_u = 162 \text{ MPa}$$  \hspace{1cm} (16)

The measured laboratory value of uniaxial compressive strength (see Table above) is 113 MPa. The coefficient of internal friction, $\mu_i$, is calculated using the second equation of (9) and the slope, $m = 2.88$, of the best fit line such that:

$$\mu_i = \frac{m - 1}{2\sqrt{m}} = 0.55$$  \hspace{1cm} (17)

The inherent shear strength, $S_0$, is related to the uniaxial compressive strength and coefficient of internal friction using (9.51):

$$S_0 = \frac{C_u}{2\left[1 + (\mu_i^2)^{1/2} + \mu_i\right]} = 48 \text{ MPa}$$  \hspace{1cm} (18)

The Coulomb criterion (9.38) is a linear relationship between the magnitude of the shear stress, $|\sigma_s|$, and the normal stress, $\sigma_n$, acting on potential shear fracture planes:

$$|\sigma_s| = S_0 - \mu_i \sigma_n = 48 \text{ MPa} - 0.55 \sigma_n$$  \hspace{1cm} (19)

c) Recall that the Mohr diagram (Figure 6.23) is a plot of the magnitude of the shear stress (ordinate) versus the normal stress (abscissa). Plot the Coulomb criterion for the Maze sandstone on a Mohr diagram. Plot the five Mohr’s half circles representing the stress states at failure from the laboratory test data. These half circles should be tangent to the line representing the Coulomb criterion if that is a good description of strength for this rock. Half circles that plot under this line would represent all possible pre-failure stress states. Because the linear Coulomb criterion does not necessarily correspond to laboratory data, Otto Mohr suggested that a smooth curve (now called the Mohr envelope) could be drawn tangent to the set of half circles from laboratory tests to define. Draw the Mohr envelope.

The MATLAB m-script `fig_09_sol_5.m` calculates the inherent shear strength and coefficient of internal friction and plots the Coulomb criterion for the Maze sandstone on a Mohr diagram.

```matlab
% fig_09_sol_5.m
% analysis of lab data for Maze sandstone
clear all, clf reset % clear functions and figures
[PCD,DCD]=textread('maze.txt','%n%n'); % read data file
PC = 0.098*PCD; DC = 0.098*DCD;
S1 = -PC; S3=-(DC+PC);
plot(S1,S3,'go'); title('Maze Sandstone');
xlabel('S1 (MPa)'), ylabel('S3 (MPa)'), axis([-250 0 -1000 0]);
MI = (2.88 - 1)/(2*sqrt(2.88));
S0 = 162/(2*(MI + sqrt(1 + MI^2)));```
d) In principle, the orientation of the shear fracture plane in a laboratory specimen should be given by the orientation of the line drawn from the center of a particular Mohr circle to the point where it touches the Mohr envelope. This line is oriented at the angle $2\gamma_C$ to the abscissa, where $\gamma_C$ is the angle between the direction of $\sigma_1$ and the normal to the fracture plane (Fig. 6). Construct such lines for the Maze sandstone deformed at 49 and 98 MPa confining pressures and determine the predicted orientation of the shear fracture. Compare your result to the shear fractures on the photographs from Hoshino et al., 1972.

The orientation of potential shear fracture planes are related to the directions of the two extreme principal stresses in Figure 6. Note that $\gamma_C$ is the angle between the direction of $\sigma_1$ and the normal to the fracture plane. For the two specimens of Maze sandstone shown on the photograph we have:

For $P_C = 49$ MPa, $2\gamma_C \approx 57^\circ$, so $\gamma_C \approx 29^\circ$  
(20)  
For $P_C = 98$ MPa, $2\gamma_C \approx 61^\circ$, so $\gamma_C \approx 31^\circ$

Note that $\gamma_C$ also is the angle between the direction of the greatest compressive stress, $\sigma_3$, (the axial compression for these triaxial tests) and the plane of the shear fracture. Measurements of the angles on the photograph yield the following:
For XC1 at $P_c = 49$ MPa $\gamma_c \approx 25^\circ$ and $28^\circ$ (21)
For XC3 at $P_c = 98$ MPa, $\gamma_c \approx 31^\circ$
There are two shear fractures visible on XC1 with somewhat different angles. Inelastic deformation for both samples was characterized as single shear fractures by Hoshino et al., 1972. The correspondence between the predicted angles using the Mohr-Coulomb concepts and the measured angles from the laboratory tests is good.

![Diagram of shear fractures](image)

Figure 6. Orientations of the two sets of potential shear fracture planes are given by the normal $n$. For the Coulomb criterion these orientations are related to the directions of the extreme principal stresses.

3) A. A. Griffith (1924) used different combinations of remote biaxial loading for the inclined elliptical hole to investigate the growth of cracks from flaws in elastic solids. The distribution of tangential stress, $\sigma_t$, on the hole wall is given by (9.65). Note that there are the three applied loads (two remote principal stresses, $\sigma_1$ and $\sigma_3$, and an internal pressure, $P$) and three geometric parameters (half-length, $a$, half-width, $b$, and inclination angle, $\beta$). Position around the hole is prescribed using the angle $\eta$. Because of the symmetry of this problem one only need consider the range $-90^\circ \leq \eta \leq +90^\circ$.

a) Begin the analysis with the case of internal pressure of unit magnitude, $P = 1$, no remote stress, and a ratio $a/b = 4$. Systematically vary the inclination of the hole by changing the angle $\beta$ from $0^\circ$ to $90^\circ$ in increments of $15^\circ$ and plot the distributions of tangential stress versus position around the hole on the same graph so they can be compared. Explain why the stress distribution does not vary with inclination of the hole. Describe the distribution, noting the magnitude and position of the maximum and minimum tangential stress.
The MATLAB m-script fig_09_sol_7.m calculates the tangential stress, $\sigma_t$, and plots this versus position on the hole boundary.

```matlab
% fig_09_sol_7.m
% calculate and plot normal stress acting tangential
% to boundary of elliptical hole (9.65)
clear all, clf reset; % clear memory and figures
s1 = 0; s3 = 0; p = 1; % remote stress and internal pressure
a = 1; b = .25; % semi-major and -minor axial lengths
ETD=-90:1:90; ETR=ETD*pi/180; % angle eta
amb = a^2 - b^2;
hold on; axis([-90 90 -2 10]);
xlabel('Eta (degrees)'), title('Elliptical Hole');
ylabel('Tangential stress/Internal pressure');
C=['b' 'g' 'r' 'c' 'm' 'y' 'k']
for k = 1:7
    be = (k-1)*15; ber = be*pi/180; % angle beta
c2b = cos(2*ber); C2E = cos(2*ETR);
DEN = (a^2 + b^2) - amb*C2E;
T1 = (s1+s3+2*p)*(2*a*b)./DEN;
T2 = (s1-s3)*(((a+b)^2)*(cos(2*(ber-ETR))) - amb*c2b)./DEN;
ST = -p + T1 + T2;
plot(ETD,ST,C(k));
end
legend('0','15','30','45','60','75','90');
```

The stress distribution (Figure 7) is sharply peaked and symmetric about $\eta = 0^\circ$. The maximum stress concentration is at $\eta = 0^\circ$ and has a value of 7.

![Figure 7](image_url)

**Figure 7.** Distribution of normal stress acting tangential to elliptical hole boundary for axial ratio, $a/b = 4$, no remote stress, and unit internal pressure.
The greatest stress diminution is at $\eta = 90^\circ$ and has a value of $-1/2$. As $\beta$ varies, changing the orientation of the hole, this stress distribution is invariant. Because the two remote principal stresses are equal the remote stress field is isotropic and therefore the orientation of the hole does not affect the stress distribution anywhere. If the remote principal stresses were unequal, the remote field would be anisotropic and the perturbed stress field due to the elliptical hole would depend upon the orientation of the hole relative to the remote field (see part b).

b) Consider the case of a uniaxial remote tension of unit magnitude, $\sigma_1 = 1$, parallel to the $y$-axis, no internal pressure, and a ratio $a/b = 4$. Systematically vary the inclination of the hole by changing the angle $\beta$ from $0^\circ$ to $90^\circ$ in increments of $15^\circ$ and plot the distributions of tangential stress versus position around the hole on the same graph so they can be compared. Describe how the stress distributions change with inclination.

With minor adjustments the MATLAB m-script fig_09_sol_7.m calculates the tangential stress distributions and plots these versus position on the hole boundary (Figure 8).

![Figure 8. Distribution of normal stress acting tangential to elliptical hole boundary for axial ratio, $a/b = 4$, uniaxial remote tension parallel to the $y$-axis, zero internal pressure, and a variety of orientations.](image)

Note that the uniaxial tension of unit magnitude is directed parallel to the $y$-axis so when $\beta = 0^\circ$ the long axis of the elliptical hole is perpendicular to the applied stress. The greatest stress concentration occurs at the end of the long axis, $\eta = 0^\circ$, with the long axis perpendicular to the applied stress, $\beta = 0^\circ$. This maximum decreases in magnitude and shifts counterclockwise around the boundary as the hole inclines. For $\beta = 90^\circ$ the maximum is at the end of the short axis, $\eta = 90^\circ$. The greatest stress diminution occurs at
the end of the short axis, $\eta = -90^\circ$, with the long axis perpendicular to the applied stress, $\beta = 0^\circ$. This minimum increases in magnitude (becomes more compressive) and shifts clockwise around the boundary as the hole inclines and then decreases in magnitude. For $\beta = 90^\circ$ the minimum is at the end of the long axis, $\eta = 0^\circ$.

c) Griffith made the surprising discovery that the flaw could induce tensile stresses even though the applied stress is compressive. Impose a uniaxial remote compression of unit magnitude, $\sigma_3 = -1$, parallel to the $x$-axis and no internal pressure. Plot and describe the distribution of tangential stress on the hole boundary for different inclinations, varying the angle $\beta$ from $0^\circ$ to $90^\circ$ in increments of $15^\circ$. Describe the range of $\eta$ over which the tangential stress is tensile. What conclusions can you draw regarding the effects of inclination on the stress distribution? From the plot estimate the critical angle of inclination, $\beta_c$, at which the tangential stress is greatest. Draw a sketch of the hole inclined at the critical angle and show where an opening fracture would initiate on the boundary and in what direction it would propagate. Use symmetry to determine where a second fracture would initiate and show its propagation direction.

With minor adjustments the MATLAB m-script fig_09_sol_7.m calculates the tangential stress distributions and plots these versus position on the hole boundary (Figure 9).

![Figure 9. Distribution of normal stress acting tangential to elliptical hole boundary for axial ratio, $a/b = 4$, uniaxial remote compression parallel to the $x$-axis, zero internal pressure, and a variety of orientations.](image)

The applied compression induces a narrowly-peaked tensile maximum at the end of the long axis, $\eta = 0^\circ$, if the long axis is parallel to the applied stress, $\beta = 0^\circ$. This tensile maximum increases in magnitude and shifts counterclockwise around the boundary as the hole inclines toward $\beta = 30^\circ$. The maximum then decreases in magnitude and the peak
broadens and shifts further counterclockwise. For $\beta = 90^\circ$ the tensile maximum is at the end of the short axis, $\eta = 90^\circ$. From this graph we estimate the critical angle of inclination, at which the tangential stress is greatest, to be about $\beta_c = 30^\circ$. In that orientation the position on the hole boundary where the tension is a maximum is about $\eta = 15^\circ$, counterclockwise from the end of the long axis. Figure 10 is a sketch of the hole in about this orientation showing an opening fracture that has propagated outward along a curving path to become aligned with the direction of applied compression. A second fracture is shown symmetrically located at about $\eta = 195^\circ$. Such fractures are commonly observed near the tips of sheared joints and are referred to as ‘wing’ cracks or ‘splay’ cracks. In a laboratory experiment on a glass block subject to uniaxial compression, wing cracks formed near the ends of three pre-cut echelon flaws (Figure 9.28).

Figure 10. Sketch of elliptical flaw loaded by uniaxial compression with predicted opening cracks that initiate at the boundary perpendicular to the greatest tensile stress.

d) Investigate the effect of changing the shape of the cavity on the tangential stress. Impose a uniaxial remote compression of unit magnitude, $\sigma_3 = -1$, parallel to the $x$-axis and no internal pressure. Choose an inclination of $\beta = 30^\circ$ and decrease the hole width, $2b$, in the sequence 0.35, 0.30, 0.25, 0.20, 0.15, 0.10, 0.05. Plot and describe how the distribution of tangential stress changes as the ratio of $b$ to $a$ decreases and the hole becomes more crack-like. Given this result and the result from part c) describe the shape and orientation of ‘most dangerous flaw’ in a rock mass subject to compression.

With minor adjustments the MATLAB m-script fig_09_sol_7.m calculates the tangential stress distributions and plots these versus position (Figure 11).

For all of the selected axial ratios, $b/a$, the applied compression induces a narrowly-peaked tensile maximum at small positive values of $\eta$. In other words the point of initiation of an opening fracture is counterclockwise from the end of the long axis as illustrated in Figure 10. The magnitude of the greatest tension increases in an apparently
non-linear way as the axial ratio decreases. The tensile maximum shifts toward the end of the long axis of the hole as the axial ratio decreases.

Given these results, and those from part c), the most dangerous flaw would be the one with the smallest axial ratio (the most crack-like in shape) that is oriented at the critical angle, $\beta_c$, which is oblique to the direction of compression. Griffith formalized the definition of the most dangerous flaw using Calculus.

![Elliptical Hole](image)

Figure 11. Distribution of normal stress acting tangential to the elliptical hole boundary for a uniaxial remote compression parallel to the $x$-axis, zero internal pressure, and a series of axial ratios, $b/a$, approaching a crack-like geometry.

4) Fractures in rock are idealized as two adjacent, mirror image surfaces that are bounded in extent along a common curved line called the tipline. The direction of the displacement discontinuity between originally-adjacent points on these surfaces near the fracture tipline serves to classify fractures into three modes. For mode-I fractures the displacement discontinuity is perpendicular to the fracture surfaces so they either open or close. Natural opening fractures include joints, veins, and dikes. In this exercise the near-tip mode-I stress field (9.72) is investigated to gain insights about the propagation of dikes and the associated damage zone.

a) For a uniform remote normal stress, $\sigma_{yy}^r$, and a uniform normal stress at the fracture surfaces, $\sigma_{yy}^c$, the mode-I stress intensity factor is

$$K_1 = \left(\sigma_{yy}^r - \sigma_{yy}^c\right)\sqrt{\pi a}$$  \hspace{1cm} (22)

For this loading state analyze the dimensions of the equations for the near-tip stress components and show that they are dimensionally homogeneous. Indicate the dimensions and units (S.I.) for the stress intensity factor.
The Cartesian components of stress near the opening fracture tip under uniform remote and fracture surface stresses are taken from (9.72) with the mode-I stress intensity defined in (22):

\[
\begin{align*}
\sigma_{xx} & = \left( \sigma_{yy}^r - \sigma_{yy}^s \right) \sqrt{\pi a} \left\{ \frac{\cos(\theta/2) \left[ 1 - \sin(\theta/2) \sin(3\theta/2) \right]}{\sqrt{2\pi r}} \right\} \\
\sigma_{yy} & = \left( \sigma_{yy}^r - \sigma_{yy}^s \right) \sqrt{\pi a} \left\{ \frac{\cos(\theta/2) \left[ 1 + \sin(\theta/2) \sin(3\theta/2) \right]}{\sqrt{2\pi r}} \right\} \\
\sigma_{xy} & = \left( \sigma_{yy}^r - \sigma_{yy}^s \right) \sqrt{\pi a} \left\{ \sin(\theta/2) \cos(\theta/2) \cos(3\theta/2) \right\}
\end{align*}
\] (23)

A dimensional analysis of (23) demonstrates that these equations are dimensionally homogeneous:

\[
ML^{-1}T^{-2} = \frac{\left( ML^{-1}T^{-2}\right) \left(L\right)^{1/2}}{\left(L\right)^{1/2}} = \left( ML^{-1}T^{-2}\right) \left(L^{0}\right) = \left( ML^{-1}T^{-2}\right)
\] (24)

Considering the numerator on the right side of the first equal sign in (24), the dimensions of stress intensity are those of stress times the square root of length:

\[
K_1 \equiv \left( ML^{-1}T^{-2}\right) \left(L\right)^{1/2} = ML^{-1/2}T^2
\] (25)

Appropriate S.I. units would be:

\[
\text{stress intensity \( = \) MPa \cdot m^{1/2}}
\] (26)

b) Calculate the Cartesian stress components for the near-tip stress field of the mode-I fracture. Recall that these functions are good approximations only for \( r < 0.01a \) where \( a \) is the half length of the fracture. Choose \( a = 1 \) and \( K_1 = 1 \) and prepare contour plots of each component. Be sure to avoid the region very close to the tip. Describe these three stress distributions, commenting on how the sign and magnitudes of each component vary spatially. Rationalize the distribution of signs for the shear stress based on the kinematics of fracture opening.

The MATLAB m-script `fig_09_sol_12.m` calculates the near-tip stress distributions for the mode-I fracture and plots these as contour maps (Figure 12).

```matlab
% fig_09_sol_12.m
% Mode I near-tip Cartesian stress components (9.72)
clear all, clf reset % clear functions and figures
k1 = 1; % mode-I stress intensity
x=-0.031:0.001:0.031; % Define x-coords. of grid
% x = x + eps; % add eps to avoid singularity at origin
y=-0.031:0.001:0.031; % Define y-coords. of grid
[X,Y]=meshgrid(x,y); % Define rectangular grid
[TH,R]=cart2pol(X,Y); % Convert Cart. to polar coord.
ST2=sin(TH/2); S3T2=sin(3*TH/2); CT2=cos(TH/2); C3T2=cos(3*TH/2);
R2P=sqrt(2*pi*R); R2P(find(R<0.0015)) = nan; % exclude near tip
SXX=k1*(CT2.*(1-ST2.*S3T2))./R2P; % xx normal stress component
SYY=k1*(CT2.*(1+ST2.*S3T2))./R2P; % yy normal stress component
SXY=k1*(ST2.*CT2.*C3T2)./R2P; % xy shear stress component
% contour plots of stress and energy components
contourf(X,Y,SXX,10), colorbar, title('mode-I stress sxx')
```
xlabel('x/a'), ylabel('y/a'), axis equal tight
figure, contourf(X,Y,SYY,10), colorbar, title('mode-I stress syy')
xlabel('x/a'), ylabel('y/a'), axis equal tight
figure, contourf(X,Y,SXY,10), colorbar, title('mode-I stress sxy')
xlabel('x/a'), ylabel('y/a'), axis equal tight

a) mode-I stress sxx

b) mode- stress syy

c) mode-I stress sxy
Figure 12. Contour maps of near-tip Cartesian stress components for mode-I fractures. The distal portion of the fracture is shown as a white line and white circles cover the stress singularity at the tip.

The stress concentration associated with the component $\sigma_{xx}$ forms a lobe in front of the tip that is symmetric about the fracture plane. Two lesser lobes lie behind the tip and are symmetrically arranged to either side of the fracture plane. The stress is tensile in all of these lobes.

The stress concentration associated with the component $\sigma_{yy}$ forms two lobes that spread out to either side of the fracture plane, mostly in front of the tip. They are symmetric about the fracture plane. The stress is tensile in both of these lobes.

The stress concentration associated with the component $\sigma_{xy}$ forms four lobes that spread out from the fracture tip occupying the four quadrants of the Cartesian coordinate system. The shear stress is positive in quadrants 1 and 3, and negative in quadrants 2 and 4. The shear stress is greater in magnitude at comparable distances from the tip in the two lobes in quadrants 2 and 3. Figure 13 illustrates the near-tip kinematics and rationalizes the distribution of signs of the shear stress shown in Figure 12c. For example, the inward motion of the fracture tip corresponds to a positive shear strain and stress in quadrant 1. The increase in opening displacements away from the tip in quadrant 2 corresponds to a negative shear strain and stress.

![Image of Figure 12](image)

Figure 13. Kinematics of opening fracture related to the sign of the strain and stress near the right-hand fracture tip.

c) Derive an equation for the maximum shear stress, $\sigma_s$, in the $(x, y)$-plane as a function of the Cartesian stress components for this plane strain problem.
Calculate \( \sigma_s \) for the near-tip field and compare a contour plot of this stress with the photoelastic image. Compare the distributions of \( \sigma_s \) and \( \sigma_{xy} \) (from part b) pointing out similarities and differences.

Recall that the two principal stresses in the plane of this plane strain problem are related to the Cartesian components as in (6.72):

\[
\sigma_1, \sigma_2 = \frac{1}{2} \left( \sigma_{xx} + \sigma_{yy} \right) \pm \left[ \frac{1}{4} \left( \sigma_{xx} - \sigma_{yy} \right)^2 + \sigma_{xy}^2 \right]^{1/2}
\]  

(27)

Also recall that the maximum shear stress in the \((x, y)\)-plane is related to the principal stresses in this plane as in Table 6.1:

\[
\sigma_s = \frac{1}{2} \left( \sigma_1 - \sigma_2 \right)
\]  

(28)

Substituting (27) into (28) we find:

\[
\sigma_s = \left[ \frac{1}{4} \left( \sigma_{xx} - \sigma_{yy} \right)^2 + \sigma_{xy}^2 \right]^{1/2}
\]  

(29)

Adding three lines of code using (29) to MATLAB m-script fig_09_sol_12.m calculates the near-tip maximum shear stress distributions for the mode-I fracture and plots this as a contour map (Figure 14).

The photoelastic image (Figure 9.20) bears a remarkable similarity to the contour map of \( \sigma_s \) in that the bands of distinct color have a very similar pattern. In other words the contour lines are similar for the theoretical and experimental cases. In both cases the stress concentration associated with the maximum shear stress forms two lobes that spread away from the tip along the \( y \)-axis and are symmetric about this axis. The photoelastic image covers a much greater area and it can be seen that the two lobes diverge from being symmetric about the \( y \)-axis with greater distance from the tip. The lobes spread outward in the positive \( x \)-direction more than in the negative \( x \)-direction.

![Mode-I Stress SS](image)

**Figure 14.** Contour map of near-tip maximum shear stress for the mode-I fracture. The distal portion of the fracture is shown as the white line and the white circle covers the stress singularity at the tip.
The contour maps of $\sigma_s$ (Figure 14) and $\sigma_{xy}$ (Figure 12c) are quite different. The Cartesian component of shear stress has four lobes each spreading outward into a different quadrants whereas the maximum shear stress has only two lobes and they spread along the $y$-axis. Furthermore, the Cartesian shear stress changes sign from positive to negative to positive to negative on a circuit around the tip (Figure 13), whereas the maximum shear stress is positive everywhere by definition.

d) Consider the distribution of ground cracks in Keanakakoi ash near a fissure that erupted from a dike on Kilauea volcano (also see Figure 9.32). Why did more than one crack form? Why are the cracks distributed to either side of the trace of the fissure? To address these questions suppose the cracks formed in the near-tip region of the dike as it approached the surface. Calculate the distributions of the normal stress components, $\sigma_{xx}$ and $\sigma_{yy}$, about an opening fracture tip by plotting the appropriate functions of $\theta$ from (9.72) versus the polar angle over the range, $-\pi \leq \theta \leq +\pi$. Describe these stress distributions and use them to predict the location and orientation of secondary fractures.

The MATLAB m-script fig_09_sol_15.m calculates the normalized near-tip Cartesian stress components for the mode-I fracture and plots these versus angle $\theta$ (Figure 15).

```matlab
% fig_09_sol_15
% Mode I near-tip Cartesian stress distributions
clear all, clf reset % clear functions and figures
TH = -pi:pi/180:pi; THD = TH*180/pi;
ST2=sin(TH/2); S3T2=sin(3*TH/2); CT2=cos(TH/2); C3T2=cos(3*TH/2);
SXX=CT2.*(1-ST2.*S3T2); % xx normal stress component
SYY=CT2.*(1+ST2.*S3T2); % yy normal stress component
SXY=ST2.*CT2.*C3T2; % xy shear stress component
plot(THD,SXX,'b'), title('near-tip stress components: mode I');
hold on, axis([-180 180 0 1.5]);
xlabel('theta (degrees)'), ylabel('normalized stress');
plot(THD,SYY,'r'), legend('sxx','syy');
```

As illustrated in Figure 9.30 the $x$-axis is in the plane of the fracture and perpendicular to the tipline, whereas the $y$-axis is perpendicular to the fracture plane and to the tipline. Therefore, for a shallow vertical dike with a sub-horizontal tipline $\sigma_{xx}$ would be vertical and $\sigma_{yy}$ would be horizontal and directed perpendicular to the plane of the dike. Directly ahead of the opening mode fracture these two Cartesian normal stress components are equal and tensile (Figure 15). To either side of that point $\sigma_{xx}$ decreases in magnitude, but $\sigma_{yy}$ increases to dual maxima with values about 30% greater than those immediately ahead of the fracture. This non-intuitive result provides an explanation for the distribution of ground cracks in the Keanakakoi ash. These cracks are sub-parallel to the trace of the fissure and therefore sub-parallel to the plane of the inferred dike. They are perpendicular to the direction of $\sigma_{yy}$ which is the greater of the two tensile stresses. Furthermore the dual maxima explain why the ground cracks are located to either side of the trace of the fissure. The stress distributions would change somewhat if the traction free surface were explicitly included, but the dual maxima for the normal stress acting parallel to the surface is preserved (Pollard et al., 1983).
5) For mode-II fractures the displacement discontinuity is parallel to the fracture surfaces and perpendicular to the tipline. Natural examples of sliding fractures include sheared joints, deformation bands, and faults. The sheared joint used for this exercise has a kinked end which is about 15 cm long, whereas the main trace of the fracture extends straight off the left side of the photograph many times this length. The trace of the kinked portion is oriented about 43° counterclockwise from the main trace.

a) To address questions about the propagation of sheared joints suppose the straight portion formed as an opening fracture and the kinked portion formed later when left-lateral shearing developed. To investigate the propagation of the joint calculate the distributions of the Polar stress components \( \sigma_{rr} \), \( \sigma_{\theta\theta} \), and \( \sigma_{r\theta} \) about an opening (mode-I) fracture tip as given by Lawn and Wilshaw (1975):

\[
\begin{align*}
\begin{cases}
\sigma_{rr} \\
\sigma_{\theta\theta} \\
\sigma_{r\theta}
\end{cases} \approx K_I 
\begin{cases}
\cos\left(\frac{\theta}{2}\right) \left[1 + \sin^2\left(\frac{\theta}{2}\right)\right] \\
\cos^3\left(\frac{\theta}{2}\right) \\
\sin\left(\frac{\theta}{2}\right) \cos^2\left(\frac{\theta}{2}\right)
\end{cases} \\
\left(\frac{2\pi r}{K_I}\right)^{1/2}
\end{align*}
\]

(30)

Normalize the stress components so they are functions of \( \theta \) only and plot these versus the polar angle over the range \(-\pi \leq \theta \leq +\pi\). Describe these stress distributions in terms of their symmetry, signs, and magnitudes. Discuss how the near-tip stress distribution might influence the direction of propagation.

The normalized Polar stress components in the near-tip region are:
The MATLAB m-script `fig_09_sol_16.m` calculates and plots the near-tip Polar stress components (Figure 16).

```matlab
% fig_09_sol_16
% Mode I & II near-tip Polar stress distributions
clear all, clf reset % clear functions and figures
TH = -pi:pi/180:pi; THD = TH*180/pi;
ST2=sin(TH/2); CT2=cos(TH/2);
SRR=CT2.*(1+(ST2.*ST2)); % rr normal stress component
STT=CT2.*CT2.*CT2; % tt normal stress component
SRT=ST2.*CT2.*CT2; % rt shear stress component
plot(THD,SRR,'b'), hold on, axis([-180 180 -.5 1.5]);
title('near-tip polar stress components: mode I');
xlabel('theta (degrees)'), ylabel('normalized stress');
plot(THD,STT,'r'), plot(THD,SRT,'g'), legend('srr','stt','srt');
SRR=-ST2.*(1-3*ST2.*ST2); % rr normal stress component
STT=3*ST2.*CT2.*CT2; % tt normal stress component
SRT=-CT2.*(1-3*ST2.*ST2); % rt shear stress component
figure, plot(THD,SRR,'b'), hold on, axis([-180 180 -2 2]);
title('near-tip polar stress components: mode II');
xlabel('theta (degrees)'), ylabel('normalized stress');
plot(THD,STT,'r'), plot(THD,SRT,'g'), legend('srr','stt','srt');
```

Figure 16. Plot of normalized mode-I Polar stress components versus position, $\theta$, in the near-tip region of a fracture.
Both normal stresses are tensile throughout the near-tip region and the shear stress is zero on the extension of the fracture plane. The normal stress distributions for the mode-I fracture tip are symmetric about the fracture plane. This symmetry is consistent with in-plane propagation of opening mode fractures such as joints and explains the remarkably straight traces of many joints in outcrop. Any incipient crack extending from the tip of the propagating joint would be perpendicular to the circumferential normal stress, \( \sigma_{\theta\theta} \), and this component has a maximum along the extension of the joint plane where \( \theta = 0 \). The dual maxima in the radial component to either side of the fracture plane (c.f. Figure 14) might influence the development of micro-cracks in the process zone, but apparently this did not lead to a deflection of the path in the example considered here. We interpret the majority of the fracture path as forming during the jointing phase of deformation.

b) To investigate the propagation of the kinked portion of the fracture calculate the distributions of the Polar stress components (\( \sigma_r \), \( \sigma_{\theta\theta} \), and \( \sigma_{r\theta} \)) about a mode-II (sliding) fracture tip as given by Lawn and Wilshaw (1975):

\[
\begin{cases}
\sigma_r \\
\sigma_{\theta\theta} \\
\sigma_{r\theta}
\end{cases} \cong \frac{K_{II}}{(2\pi r)^{1/2}} \begin{cases}
sin(\theta/2)[1-3\sin^2(\theta/2)] \\
-3\sin(\theta/2)\cos^2(\theta/2) \\
\cos(\theta/2)[1-3\sin^2(\theta/2)]
\end{cases}
\]

These functions are for a right-lateral sense of sliding. Modify (32) to account for left-lateral sliding and to normalize the stress components as functions of \( \theta \) only. Plot the appropriate functions of \( \theta \) versus the polar angle over the range \(-\pi \leq \theta \leq +\pi\) and describe these stress distributions in terms of their symmetry, signs, and magnitudes. Discuss how this near-tip stress distribution might influence the direction of propagation.

The normalized stress components in the near-tip region of the left-lateral mode-II fracture are:

\[
\begin{cases}
\sigma_r \\
\sigma_{\theta\theta} \\
\sigma_{r\theta}
\end{cases} \cong \frac{(2\pi r)^{1/2}}{K_{II}} \begin{cases}
-\sin(\theta/2)[1-3\sin^2(\theta/2)] \\
3\sin(\theta/2)\cos^2(\theta/2) \\
-\cos(\theta/2)[1-3\sin^2(\theta/2)]
\end{cases}
\]

The MATLAB m-script fig_09_sol_16.m calculates and plots the near-tip Polar stress components (Figure 17).

The greatest magnitudes of radial stress (2.000) occur on the fracture surfaces and this stress component is directed parallel to the surfaces in these locations. The radial stress is compressive on the lower surface where \( \theta = -\pi \) and is tensile on the upper surface where \( \theta = +\pi \). The greatest magnitudes of circumferential stress (1.155) occur at \( \theta \approx \pm12^\circ \) and this stress component is compressive for the negative angle and tensile for the positive angle. It is interesting to note that the greatest circumferential stress is only about 57% of the greatest radial stress. Both radial and circumferential normal stress components are exactly zero along the extension of the fracture plane where \( \theta = 0^\circ \). The shear stress
varies in a sinusoidal manner from zero on the fracture surfaces to maxima at $\theta \approx \pm 124^\circ$ and a minimum along the extension of the fracture plane.

![Image of near-tip polar stress components: mode II](image)

Figure 17. Plot of normalized mode-II Polar stress components versus position, $\theta$.

In sharp contrast to the mode-I fracture (Figure 16), the normal stress distributions for the mode-II fracture are asymmetric about the fracture plane (Figure 17) and the shear stress magnitude is greatest on the extension of the fracture plane. The tensile maximum at $\theta \approx 71^\circ$ has been used by some to explain the development of so-called splay or wing cracks near the terminations of fractures that initially developed in the opening mode and later were sheared (e.g., Pollard & Aydin, 1988). The shearing introduces mode-II and a concentration of tensile stress out of the plane of the fracture. This could lead to the redirection of propagation at an oblique angle.

The tensile maximum in the radial component is greater in magnitude than that in the circumferential component. If the radial component were cause a redirection of propagation it would start just behind the tip and be oriented perpendicular to the fracture plane such that $\theta \approx 90^\circ$. Note that the example sheared joint has a kinked portion that is oriented at about $43^\circ$ from the main trace which is a lower angle than either predicted with this simple analysis. Analyses that consider non-uniform distributions of tractions on the fracture surfaces demonstrate that the angle and position of the incipient kinked fracture are very sensitive to the traction distribution (Willemse & Pollard, 1998). For the pure mode-II fracture with a cohesive end zone the predicted angle is $45^\circ$.

c) Explain why the kinked portion of the fracture is so short relative to the main fracture trace. Predict what you would observe at the other end of the sheared joint if your explanation for the propagation mechanics is correct.
The stress concentration associated with fractures is localized to a small region near the tip. For all modes the magnitude of all stress components decay as $1/\sqrt{r}$ with distance from the tip (30)(32). Therefore, if sliding along a pre-existing joint is the cause of fracture propagation in a new direction, one would expect that propagation to be limited in extent. In cases where kinked portions of sheared joints are long the remote stress conditions must have been conducive for opening fracture propagation in the new direction.

If kinked fractures result from sliding along a fracture that propagated to its current length in pure opening mode, there is a direct correspondence between the sense of sliding and the redirection of propagation. Left-lateral sliding is associated with a kink to the left as one looks toward the fracture tip and right-lateral sliding is associated with a kink to the right. For the example of the sheared joint one would expect to find a kinked portion at the other end that is directed to the left as one looks toward that end.