Exercise solutions: concepts from chapter 8

1) The following exercises explore elementary concepts applicable to a linear elastic material that is isotropic and homogeneous with respect to the elastic properties.

   a) It is commonly understood that longitudinal deformation, say shortening, implies compressive normal stress acting in the direction of this strain. Use the three dimensional form of Hooke's Law for an isotropic body with Young's modulus, $E$, and Poisson’s ratio, $\nu$, as the two elastic moduli to demonstrate that this could be a misconception under some states of stress. As an illustrative example consider the state of uniaxial tension, $\sigma_{xx} > 0$, $\sigma_{yy} = 0 = \sigma_{zz}$. Describe how your result depends upon the elastic moduli and define the full range of these quantities that you are considering.

For the isotropic elastic material the longitudinal strain components are related to the normal stress components as (8.12):

$$
\varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu \left( \sigma_{yy} + \sigma_{zz} \right) \right]
$$

$$
\varepsilon_{yy} = \frac{1}{E} \left[ \sigma_{yy} - \nu \left( \sigma_{zz} + \sigma_{xx} \right) \right]
$$

$$
\varepsilon_{zz} = \frac{1}{E} \left[ \sigma_{zz} - \nu \left( \sigma_{xx} + \sigma_{yy} \right) \right]
$$

(1)

Young’s modulus, $E$, and Poisson’s ratio, $\nu$, are non-negative quantities by definition. Therefore, we consider the range $0 \leq E \leq \infty$ for Young’s modulus where the end member cases refer to zero longitudinal stiffness and infinite longitudinal stiffness. Also, we note that Poisson’s ratio is restricted to the range $0 \leq \nu \leq 1/2$ by definition where the end member cases refer to perfectly compressible and perfectly incompressible. Now solve for the longitudinal strains by substituting the state of uniaxial stress into (1):

$$
\varepsilon_{xx} = \frac{1}{E} \sigma_{xx}, \quad \varepsilon_{yy} = -\frac{\nu}{E} \sigma_{xx}, \quad \varepsilon_{zz} = -\frac{\nu}{E} \sigma_{xx}
$$

(2)

For zero longitudinal stiffness the strains are all undefined and for infinite longitudinal stiffness the strains are all zero, so neither of these end member cases is of any practical importance. Taking the range $0 < E < \infty$ (2) may be evaluated such that:

$$
\varepsilon_{xx} = \frac{1}{E} \sigma_{xx} > 0, \quad \varepsilon_{yy} = -\frac{\nu}{E} \sigma_{xx} \leq 0, \quad \varepsilon_{zz} = -\frac{\nu}{E} \sigma_{xx} \leq 0
$$

(3)

If Poisson’s ratio is greater than zero, the body shortens in the $y$ and $z$ directions although the normal stress components in these directions are zero (not compressive).
One could add a tension in the $y$ or $z$ direction of magnitude $<\nu\sigma_{xx}$ and the body would still shorten in that direction. Thus, the applied normal stress can be tensile in the direction that the body shortens.

b) Now consider a three-dimensional state of stress that could develop in Earth’s crust. The stresses are given by Anderson’s standard state (an isotropic compression). We ignore strains that are associated with the development of this stress state. Suppose the rock body is subject to a tectonic stress state $\sigma_{xx} > 0, \sigma_{yy} = \sigma = \sigma_{zz},$ and $\sigma > 0.$ In other words a tectonic tension is applied in the $x$-direction and a tension of magnitude $\sigma$ is applied in the $y$- and $z$-direction. Use Hooke’s Law for an isotropic body with Young’s modulus, $E,$ and Poisson’s ratio, $\nu,$ as the two elastic moduli to determine those conditions under which the tectonic strains in $y$ and $z$ are a shortening even though the tectonic stress is tensile in those directions.

Solving for the longitudinal strains by substituting the given state of tectonic stress into (1) we have:

$$\varepsilon_{xx} = \frac{1}{E} [\sigma_{xx} - 2\nu\sigma], \quad \varepsilon_{yy} = \frac{1}{E} [\sigma - \nu(\sigma + \sigma_{xx})] = \varepsilon_{zz}$$

Rearranging the second equation and calling this longitudinal strain $\varepsilon$, we find:

$$\varepsilon = \varepsilon_{yy} = \varepsilon_{zz} = \frac{1}{E} [\sigma (1 - \nu) - \nu \sigma_{xx}]$$

This longitudinal strain is a shortening (negative) under the following conditions:

$$\varepsilon < 0 \text{ if } \sigma < \frac{\nu}{1 - \nu} \sigma_{xx}$$

That is, the rock shortens in $y$ and $z$ if the tectonic tensile stress in those directions is less than $\nu/(1 - \nu)$ times the tectonic tensile stress in the $x$-direction. For a typical value of Poisson’s ratio, say $\nu = 1/4$, the tectonic tensile stress in $y$ and $z$ would have to be less than one third the tectonic stress in the $x$-direction.

c) It is commonly understood that shearing deformation implies shear stresses acting on planes associated with this strain. Use the three dimensional form of Hooke’s Law for an isotropic body with Young’s modulus and Poisson’s ratio as the two elastic moduli to demonstrate that this is an accurate conception. Describe how your result depends upon the elastic moduli and define the full range of these quantities that you are considering.

The shear strain components are related to the shear stress components as (8.17):

$$\varepsilon_{xy} = \frac{1 + \nu}{E} \sigma_{xy}, \quad \varepsilon_{yx} = \frac{1 + \nu}{E} \sigma_{yx}, \quad \varepsilon_{zx} = \frac{1 + \nu}{E} \sigma_{zx}$$

Young’s modulus, $E$, and Poisson’s ratio, $\nu$, are non-negative quantities by definition. For zero Young’s modulus the strains are all undefined and for infinite Young’s modulus the strains are all zero, so neither of these cases is of practical importance. Therefore we consider the ranges $0 < E < \infty$ and $0 \leq \nu \leq 1/2.$ Regardless of the values in the
permissible ranges, a non-zero shear strain is always associated with a non-zero shear stress of the same sign.

d) Many types of rubber have values of Poisson’s ratio approaching the upper limit of 1/2, whereas many varieties of cork have values approaching the lower limit of 0. Both of these materials are used as stoppers for bottles containing liquids. What mechanical reason can you offer for the predominant usage of cork instead of rubber for wine bottle stoppers? Assume that the stopper would be a solid cylindrical shape whether rubber or cork. On the other hand, rubber is the choice for most stoppers in a chemistry lab, presumably because of its resistance to chemical reaction. Can you suggest why most of these rubber stoppers are tapered and not cylindrical in shape?

From a mechanical perspective material with a lesser value of Poisson’s ratio is easier to insert into the narrow neck of a wine bottle. When the stopper is pushed into the neck of the bottle the application of a compressive stress along the cylindrical axis causes the stopper to expand in the radial direction. The less the value of Poisson’s ratio, the less expansion, and therefore the easier it is to push the stopper into the neck of the bottle. The rubber stoppers are tapered because the significant radial expansion, due to great Poisson’s ratio, would prevent them from being pushed into the neck of the bottle.

e) Suppose a rock mass has a Young’s modulus, \( E = 25 \) GPa, and Poisson's ratio, \( \nu = 0.15 \). Determine values for Lamé's constants, \( G \) and \( \lambda \), and the bulk modulus, \( K \), and write down the equations you have used. Suppose you know values for the bulk modulus, \( K \), and the shear modulus, \( G \). Using algebra, derive equations for Young's modulus, \( E \), and Poisson's ratio, \( \nu \).

The first Lamé's constant is the elastic shear modulus (8.26):
\[
G = \frac{E}{2(1+\nu)} = 11 \text{ GPa}
\] (8)

The first Lamé's constant is (8.27):
\[
\lambda = \frac{E\nu}{(1+\nu)(1-2\nu)} = 4.7 \text{ GPa}
\] (9)

The bulk modulus is (8.25):
\[
K = \frac{E}{3(1-2\nu)} = 12 \text{ GPa}
\] (10)

To derive the equation for \( E \) in terms of \( K \) and \( G \) first solve (8) and (10) for \( \nu \):
\[
1 + \nu = \frac{E}{2G} \quad \text{so} \quad \nu = \frac{E}{2G} - 1, \quad 1 - 2\nu = \frac{E}{3K} \quad \text{so} \quad \nu = \frac{1}{2} - \frac{E}{6K}
\] (11)

Then eliminate \( \nu \) to find:
\[
\frac{E}{2G} - 1 = \frac{1}{2} - \frac{E}{6K}
\] (12)

Rearranging (12) to isolate \( E \) we have:
\[ E \left( \frac{1}{2G} + \frac{1}{6K} \right) = \frac{3}{2} \text{ so } E = \frac{3}{\left( \frac{1}{G} + \frac{1}{3K} \right)} \]  

(13)

Multiplying the right side of (13) by \(3KG/3K\) the required result is:

\[ E = \frac{9KG}{3K + G} \]  

(14)

To derive the equation for \(\nu\) in terms of \(K\) and \(G\) first solve (8) and (10) for \(E\):

\[ E = 2G(1 + \nu) \text{ and } E = 3K(1 - 2\nu) \]  

(15)

Then eliminate \(E\) to find:

\[ 2G + 2G\nu = 3K - 6K\nu \]  

(16)

Rearranging (16) to isolate \(\nu\) we have:

\[ \nu(6K + 2G) = 3K - 2G \]  

(17)

Finally, solving (17) for \(\nu\) the required result is:

\[ \nu = \frac{3K - 2G}{2(3K + G)} \]  

(18)

2) Consider a block of rock that is linear elastic and isotropic and homogeneous with respect to elastic properties. Suppose the elastic properties of this block are Young’s modulus, \(E = 50\) GPa, and Poisson’s ratio, \(\nu = 0.20\). Also suppose the side lengths \(B = 1000\) m, \(H = 150\) m, and \(W = 150\) m.

a) Compute the three infinitesimal longitudinal strain components (\(\varepsilon_{xx}, \varepsilon_{yy}, \varepsilon_{zz}\)) in the coordinate directions within this block for the following state of stress:

\[ \sigma_{xx} = -50 \text{ MPa, } \sigma_{yy} = -35 \text{ MPa, } \sigma_{zz} = -30 \text{ MPa} \]

\[ \sigma_{xy} = \sigma_{yx} = \sigma_{xz} = 0 \]  

(19)

Note that the normal components are principal stresses and all are compressive. Assess the magnitude of the strains are indicate if they are within the range for typical elastic behavior.

Using (1) to compute the components of strain we find, for example:

\[ \varepsilon_{xx} = \frac{1}{E} \left[ \sigma_{xx} - \nu \left( \sigma_{yy} + \sigma_{zz} \right) \right] = \frac{1}{5 \times 10^4} \left[ -50 + 0.2 \left( -35 - 30 \right) \right] \]  

(20)

Evaluating (20) and carrying out similar calculations for the other components we find:

\[ \varepsilon_{xx} = -7.4 \times 10^{-4}, \varepsilon_{yy} = -3.8 \times 10^{-4}, \varepsilon_{zz} = -2.6 \times 10^{-4} \]  

(21)

All of the longitudinal strains are contractions and all are within typical limits for elastic behavior.

b) Recall the general kinematic equations relating the infinitesimal strain components to the displacement components (5.118):

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

(22)
Compute the displacement components \((u_x, u_y, u_z)\) at the point \(x = 1000\) m, \(y = 150\) m, and \(z = 125\) m for the stress state given in (19). Assume the rock mass is fixed (zero displacement) at the origin of the coordinate system.

Solving (22) for the \(x\)-component of longitudinal strain:

\[
\varepsilon_{xx} = \frac{\partial u_x}{\partial x}, \text{ but } u_x = u_x(x) \text{ only, so } \varepsilon_{xx} = \frac{du_x}{dx} \tag{23}
\]

The fact that the displacement component in \(x\) is only a function of \(x\) follows from the symmetry of the loading: there are no shear stresses acting in the coordinate planes which therefore are principal stress planes. Thus, the partial derivative may be written as an ordinary derivative and the displacement is found as follows:

\[
du_x = \varepsilon_{xx}dx, \text{ so } \int du_x = \int \varepsilon_{xx}dx, \text{ and } u_x = \varepsilon_{xx}x + C_1 \tag{24}
\]

Employing the boundary conditions:

\[
\text{B.C.: } u_x = 0 \text{ at } x = 0, \text{ so } C_1 = 0 \tag{25}
\]

By similar arguments we find that each displacement component at a particular point \((B, H, W)\) relative to the origin of coordinates is given by the longitudinal strain in that direction times the distance from the origin:

\[
\begin{align*}
    u_x &= \varepsilon_{xx}B = \left(-7.4\times10^{-4}\right)(1\times10^3\text{ m}) = -7.4\times10^{-1}\text{ m} \\
    u_y &= \varepsilon_{yy}H = \left(-3.8\times10^{-4}\right)(1.5\times10^2\text{ m}) = -5.7\times10^{-2}\text{ m} \\
    u_z &= \varepsilon_{zz}W = \left(-2.6\times10^{-4}\right)(1.5\times10^2\text{ m}) = -3.9\times10^{-2}\text{ m}
\end{align*} \tag{26}
\]

All of the displacement components are negative and therefore are directed toward the origin. The displacement components at the point \((B, H, W)\) are several centimeters in the \(y\)- and \(z\)-directions and nearly \(\frac{3}{4}\) of a meter in the \(x\)-direction.

c) The stretch, \(S\), the extension (also called the infinitesimal strain), \(\varepsilon\), and the strain (also called the Lagrangian strain), \(E\), are related to the initial length, \(B\), and final length, \(b\), of the block as follows:

\[
S = \frac{b}{B}; \quad \varepsilon = \frac{b-B}{B}; \quad E = \left(\frac{b-B}{B}\right) + \frac{1}{2}\left(\frac{b-B}{B}\right)^2 \tag{27}
\]

Calculate the final length of the block, \(b\), and use this to calculate all three measures of deformation in (27). Compare the extension and the strain to determine the error introduced when using the infinitesimal strain approximation. Assess whether you were justified in using the infinitesimal theory in parts a) and b) of this exercise.

Recalling that the initial length of the block is \(B = 1000\text{ m}\) and using the first of (26) and the second of (27), the final length of the block is:

\[
b = B + u_z = 1000\text{ m} - 0.74\text{ m} = 999.26\text{ m} \tag{28}
\]

The three measures of deformation are:

\[
S = 9.9926\times10^{-1}; \quad \varepsilon = -7.4000\times10^{-4}; \quad E = -7.3972\times10^{-4} \tag{29}
\]
We define the error introduced in using the extension instead of the strain as:

\[
\text{error} = \frac{(E - \varepsilon)}{E} \times 100 = 0.04\%
\]  

(30)

To evaluate the error, values for \(E\) and \(\varepsilon\) were taken from (29). This is a very small error compared to typical measurement errors. The approximation necessary to use the extension (infinitesimal strain) instead of the strain is not significant for parts a) and b).

d) Show algebraically how the extension and strain are calculated as functions of the stretch. Use MATLAB to plot both the extension and the strain versus the stretch over the range \(0 \leq S \leq 3\). Comment on their graphical relationship to one another.

Using the first two of (27) the relationship between the extension and the stretch is:

\[
\varepsilon = \frac{b}{B} - 1 = S - 1
\]  

(31)

Using the first and third of (27) the relationship between the normal strain and stretch is:

\[
E = \frac{bB - B^2 + \frac{1}{2}(b^2 - 2bB + B^2)}{B^2} = \frac{1}{2}(b^2 - B^2) = \frac{1}{2}(S^2 - 1)
\]  

(32)

The MATLAB m-script fig_08_sol_1.m was used to plot Figure 1.

% fig_08_sol_1.m
% plot extension and strain vs. stretch
clear all, clf reset; % clear memory and figures
S = 0:0.01:3; % range of stretch
e = S - 1; % the extension
E = e + 0.5*(e.^2); % the strain
plot(S,e,'r-','.S,E','g-'), legend('extension','strain')
xlabel('stretch'), ylabel('measures of deformation');

Figure 1. The extension and the strain are plotted versus the stretch to illustrate over what range the extension is a good approximation for the strain.
The non-linear plot of the strain diverges from the linear plot of the extension for values of the stretch different from one. Note that as the stretch goes to zero the extension goes to -1 while the strain goes to -0.5:

e) Determine the approximate range of $S$ within which the error introduced by neglecting the higher order term in equation (27) for $E$ is less than 10%. Use MATLAB to plot the error as a percentage versus the stretch.

The extension (the infinitesimal strain) is an approximation for the strain (Lagrangian strain) in which the higher order term in equation (27) for $E$ is neglected. The error is defined in (30). The MATLAB m-script fig_08_sol_2.m was used to plot Figure 2.

```matlab
% fig_08_sol_2.m
% plot error for extension relative to strain
clear all, clf reset; % clear memory and figures
S = 0:0.01:3; % range of stretch
e = S-1; % the extension
E = e + 0.5*(e.^2); % the strain
error = 100*abs((E - e)./E); % calculate the error
plot(S,error,'r-'), xlabel('stretch'), ylabel('error (%)');
```

Figure 2. The error introduced when using the extension instead of the strain is plotted as a percentage versus the stretch.

The numerical results used to plot Figure 2 enable one to draw the following conclusion:

for $0.82 < S < 1.22$, error < 10%

(33)

3) The quasi-static linear elastic solution for the edge dislocation originally found application to problems of plasticity at the scale of defects in the crystal lattice. As Weertman and Weertman (1964) point out, the dislocation solution has found application as a modeling tool for many different geological structures.
Exercise solutions: concepts from chapter 8

a) Describe in words and with a carefully labeled sketch what is meant by the following attributes of the edge dislocation: extra half plane of atoms, Burgers vector, dislocation line, tangent vector, glide plane, dislocation core.

The extra half plane of atoms is coincident with \( (x = 0, \ y \leq 0, \ -\infty \leq z \leq +\infty) \). The dislocation line is the edge of this plane along the \( z \)-axis. The Burgers vector measures the distance between the starting atom, \( s \), and the finishing atom, \( f \), of a circuit around the dislocation with equal numbers of steps along each lattice plane in a plane perpendicular to the dislocation line. The tangent vector lies along the dislocation line and points in the positive \( z \)-direction. The glide plane is coincident with \( (x \leq 0, \ y = 0, \ -\infty \leq z \leq +\infty) \) and marks the displacement discontinuity. The dislocation core is the cylindrical region centered on the dislocation line with radius about five times the Burgers vector within which the deformation is inelastic.

b) Consider equations (8.36) and (8.37) which give the displacement components that solve Navier’s equations of motion for the edge dislocation. Put these equations in dimensionless form and plot each term as a function of the polar angle, \( \theta \), for a circuit around the dislocation, keeping the radial coordinate, \( r \), constant. Justify your choice of \( r \) based upon the size of the dislocation core. Identify which term(s) contribute to the displacement discontinuity across the glide plane and show how this is related to the magnitude, \( b \), of Burgers vector.

The dimensionless forms of the displacement component equations for the edge dislocation are:

\[
\frac{u_x}{b} = -\frac{1}{2\pi} \left[ \tan^{-1} \left( \frac{y}{x} \right) + \frac{G + \lambda}{2G + \lambda} \left( \frac{xy}{x^2 + y^2} \right) \right]
\]  \hspace{1cm} (34)
\[
\frac{u_v}{b} = \frac{1}{2\pi} \left[ -\left( \frac{G}{2(2G+\lambda)} \right) \ln \left( \frac{x^2 + y^2}{C} \right) + \left( \frac{G+\lambda}{2G+\lambda} \right) \left( \frac{y^2}{x^2 + y^2} \right) \right]
\]  

(35)

Note that each equation contains two terms so there are a total of four terms to plot. The MATLAB m-script `fig_08_sol_4.m` was used to plot Figure 4.

% `fig_08_sol_4.m`
% plot four terms for the normalized displacements, ux/b and uy/b,
% on a circuit around the edge dislocation at r = 10b
clear all, clf reset; % clear memory and figures
r = 10; % radius of the circuit
mu = 30000; pr = 0.25; lm = (2*mu*pr)/(1-2*pr); % Elastic moduli
c1 = (0.5*mu)/(1m+2*mu); c2 = (lm+mu)/(lm+2*mu);
TH = 0:pi/360:2*pi; THD = TH*180/pi; % Angle theta
[XC,YC] = pol2cart(TH,r); % Convert polar to Cartesian coordinates
UX1 = -(1/(2*pi))*atan2(YC,XC);
UX2 = -(1/(2*pi))*c2*(XC.*YC)./(XC.^2+YC.^2);
UY1 = -(1/(2*pi))*(-c1).*log(XC.^2+YC.^2);
UY2 = -(1/(2*pi))*c2*(YC.^2)./(XC.^2+YC.^2);
plot(THD,UUX1,THD,UX2,THD,UY1,THD,UY2);
axis([0 360 -.6 .6]), xlabel('theta (degrees)');
ylabel('normalized displacement terms, u/b');
legend('ux term 1','ux term 2','uy term 1','uy term 2');

Figure 4. Plot of each term in (34) and (35) for the displacement components, normalized by the Burgers vector, for a radial distance from the edge dislocation \( r = 10b \).

Estimating the shear strength of solids as \( G/30 \) where \( G \) is the elastic shear modulus, the shear stress near the edge dislocation would exceed this value within a radius of \( r = 5b \). To be well outside this dislocation core we use \( r = 10b \) for the plots of the terms in the elastic displacement component equations (Figure 4). Note that the positive y-axis for these plots is downward, so the angle is turned in a clockwise direction. Given the polar coordinates for points on a circuit around the dislocation (Figure 3), the Cartesian coordinates are:
\[ x = r \cos \theta, \quad y = r \sin \theta \]  

(36)

Only the first term in the equation for \( u_x \) is discontinuous (blue line in Figure 4). This term is zero on the positive \( x \)-axis where \( \theta = 0 \), and it decreases linearly toward -0.5 as the angle \( \theta \) increases and approaches \( \pi \). On the other hand turning the angle counterclockwise from the positive \( x \)-axis where \( \theta = 2\pi \), this term increases linearly toward +0.5 as the angle \( \theta \) decreases and approaches \( \pi \). Therefore, there is a discontinuity in the \( x \)-component of displacement as one crosses the glide plane (the negative \( x \)-axis).

The magnitude of the displacement discontinuity is:

\[ \Delta u_x = u_x(\theta = \pi^+) - u_x(\theta = \pi^-) = b \]  

(37)

c) Plot a contour map of each displacement component, \( u_x \) and \( u_y \), around the edge dislocation centered in a region that is 200\( b \) on a side. Choose elastic moduli such that \( G = \lambda = 3 \times 10^4 \) MPa. Compare and contrast your contour plots to those of Hytch et al. (2003) from the frontispiece for chapter 8.

The Matlab m-script fig_08_sol_56 was used to plot Figures 5 and 6.

```matlab
% fig_08_sol_56
% Displacement components, ux/b and uy/b, near edge dislocation
b = 1; % Burgers vector
mu = 30000; lm = 30000; % Elastic moduli
c1 = (0.5*mu)/(lm+2*mu); c2 = (lm+mu)/(lm+2*mu);
x = linspace(-100,100,100)+eps;
y = linspace(-100,100,100);
[X,Y]=meshgrid(x,y); % define Cartesian grid
DEN = X.^2 + Y.^2;
UXDB = (-1/(2*pi))*(atan2(Y,X) + c2*(X.*Y)./DEN);
UYDB = (-1/(2*pi))*(-c1*log(DEN) + c2*(Y.^2)./DEN);
[T,R] = cart2pol(X,Y);
UXDB(find(R<(5*b))) = nan; UYDB(find(R<(5*b))) = nan;
contourf(X,Y,UXDB,10), axis equal ij tight, colormap(jet);
xlabel('x/b'), ylabel('y/b'), title('displacement ux/b'), colorbar;
figure, contourf(X,Y,UYDB,10), axis equal ij tight, colormap(jet);
xlabel('x/b'), ylabel('y/b'), title('displacement uy/b'), colorbar;
```

Comparing Figure 5 with frames (a) and (b) of the Frontispiece for Chapter 8 (p. 287) it is clear that the contours have the same radial symmetry as the experimental plot and the theoretical plot from Hytch et al. (2003). It is not possible to compare the magnitudes of the \( x \)-component of displacement which may differ somewhat because the elastic solution used by Hytch et al. (2003) is for an anisotropic material and the solution used in the m-script fig_08_sol_56 is for an isotropic material. The Hytch plot has a discontinuity along \( y=0 \) for both \( x<0 \) and \( x>0 \). This is not consistent with a glide plane only for \( x<0 \).

Comparing Figure 6 with frames (c) and (d) of the Frontispiece for Chapter 8 (p. 287) a similar conclusion is reached. The isotropic solution for the \( y \)-component of displacement has a characteristic ‘bowtie’ pattern that is symmetric about the two Cartesian axes and this pattern is clearly seen in both the experimental plot and the theoretical plot from Hytch et al. (2003). Apparently the elementary edge dislocation solution captures the symmetry and pattern of the displacement distribution around an actual dislocation.
d) Plot a contour map of the normal stress component $\sigma_{xx}$ using the same elastic moduli and region as in part c). Describe the distribution of this stress component, pointing out any symmetry. Provide a physical explanation why this normal stress is tensile (positive) for $y > 0$ and compressive (negative) for $y < 0$.  

Figure 5. Contour map of the displacement component $u_x$ near the edge dislocation in an isotropic elastic material. The values were computed using (34). 

Figure 6. Contour map of the displacement component $u_y$ near the edge dislocation in an isotropic elastic material. The values were computed using (35).
The Matlab m-script fig_08_sol_78 was used to plot Figures 7 and 8.

```matlab
% fig_08_sol_78
% Edge dislocation - Cartesian stress components sxx, sxy
b = 1; mu = 30000; lm = 30000; % Burgers vector, elastic moduli
c1 = b/(2*pi); c2 = 2*mu*(mu+lm)/(2*mu+lm);
x = linspace(-100,100,100)+eps;
y = linspace(-100,100,100);
[X,Y]=meshgrid(x,y); % define Cartesian grid
DEN = (X.^2 + Y.^2).^2; % calculate stress components
SXX = c1*c2*((Y.*(3*X.^2 + Y.^2))./DEN);
SXY = -c1*c2*((X.*(X.^2 - Y.^2))./DEN);
[T,R] = cart2pol(X,Y); % define polar coordinates
SXX(find(R<(5*b))) = nan; SXY(find(R<(5*b))) = nan; % avoid core
contourf(X,Y,SXX,25), axis equal ij tight, colormap(jet);
title('stress sxx (MPa)'), xlabel('x/b'), ylabel('y/b'), colorbar;
figure, contourf(X,Y,SXY,25), axis equal ij tight, colormap(jet);
title('stress sxy (MPa)'), xlabel('x/b'), ylabel('y/b'), colorbar;
```

Figure 7. Contour map of normal stress component $\sigma_{xx}$ near the edge dislocation. The values were computed using (8.41).

Recall that the edge dislocation may be thought of as resulting from the insertion of an extra half plane of atoms coincident with $(x = 0, \ y \leq 0, \ -\infty \leq z \leq +\infty)$. The insertion of this additional material pushes the adjacent material outward in the positive and negative $x$-directions, thus inducing a compressive (negative) stress for $y < 0$ that is symmetric about the $y$-axis (Figure 7). The wedging action of this inserted material on the region immediately below the dislocation line induces a tensile (positive) stress for $y > 0$ that also is symmetric about the $y$-axis. Perhaps it is surprising that the distributions of this normal stress are mirror images in form across the $x$-axis, but opposite in sign, such that there is anti-symmetry about the $x$-axis.
e) Plot a contour map of the shear stress component $\sigma_{xy}$ using the same elastic moduli and region as in part c). Describe the distribution of this stress component, pointing out any symmetry. Provide a physical explanation why the shear stress is positive for $x < 0$ and negative for $x > 0$ along the $x$-axis. Indicate how this stress distribution promotes further dislocation glide.

![Contour map of shear stress component $\sigma_{xy}$](image)

Figure 8. Contour map of shear stress component $\sigma_{xy}$ near the edge dislocation. The values were computed using (8.43).

The shear stress distribution (Figure 8) is symmetric about the $x$-axis and anti-symmetric about the $y$-axis such that the major lobes of this pattern are positive along the glide plane ($x < 0, y = 0$) and negative ahead of the glide plane ($x > 0, y = 0$). The sign of the shear stress within these lobes is understood using a lattice model (Figure 9) where distortion indicates the shear strain associated with insertion of the extra half plane of atoms.

![Lattice model](image)

Figure 9. Lattice model used to understand the shear stress distribution.
Lattice distortion to the left of the dislocation line is top to the left (positive), whereas to the right of the dislocation line it is top to the right (negative). To achieve this distortion the dislocation moved along the glide plane from left to right, driven by a negative shear stress applied remotely. The dashed lines indicate bonds that were broken during the glide of the dislocation to its current position. Local negative shear stress is concentrated along the positive x-axis where it promotes further glide of the dislocation to the right.

f) Use the results from parts d) and e) to explain why the arrangement of dislocation pairs in Figure 10 can be used as a model for a right-lateral strike-slip fault.

![Figure 10. Model for a right lateral strike slip fault using a pair of edge dislocations.](image)

The glide planes for both dislocations in Figure 10 extend to the left of the respective dislocation lines (as in Figure 3). The left dislocation is associated with a left lateral offset (-b) across its glide plane whereas the offset is right lateral (+b) for the right dislocation. Therefore, the offsets cancel to the left of the left dislocation. A right lateral offset (+b) remains between the dislocations over the plane segment defined by \(|x| \leq a, \ y = 0, \ -\infty \leq z \leq +\infty\).

The model fault extends between the dislocations with a uniform slip. The relative motion, as indicated by the two arrows, would induce a compression across the leg of the T symbol that is consistent with the regions of compression (quadrants 2 and 4) for a right-lateral fault. Similarly this relative motion would induce a tension on the opposite side of the T symbol (quadrants 1 and 3). This anti-symmetric stress field is consistent with observations of secondary structures associated with strike-slip faults.

4) Consider two-dimensional, plane strain conditions defined using the following constraints on the displacement components:

\[
u_x = u_x(x, y), \quad u_y = u_y(x, y), \quad u_z = 0
\]  

(38)

In other words the two displacement components in the \((x, y)\)-plane are only functions of \(x\) and \(y\), and the \(z\)-component of displacement is zero. In this context explore the equations relating stress, strain, and displacement components, as well as the governing equations for the elastic boundary value problem.

a) Start with the three-dimensional form of Hooke’s Law for the isotropic elastic material using Lamé’s constants with stress components as dependent variables:

\[
\sigma_{ij} = 2G\varepsilon_{ij} + \lambda\varepsilon_{kk} \delta_{ij}
\]  

(39)

Using the constraints imposed by (38), and the kinematic equations (22), expand (39) for each of the six Cartesian stress components as functions of the strain components.
From (39) the out-of-plane normal stress is:

\[ \sigma_{zz} = 2G\varepsilon_{zz} + \lambda(\varepsilon_{xx} + \varepsilon_{yy} + \varepsilon_{zz}) \]  

(40)

The kinematic equations relating infinitesimal strain and displacement components are:

\[ \varepsilon_{ij} = \frac{1}{2} \left( \frac{\partial u_i}{\partial x_j} + \frac{\partial u_j}{\partial x_i} \right) \]  

(41)

Considering the longitudinal strain in the \( z \)-direction, from (41) we have:

\[ \varepsilon_{zz} = \frac{\partial u_z}{\partial z} = 0, \text{ so } \sigma_{zz} = \lambda(\varepsilon_{xx} + \varepsilon_{yy}) \]  

(42)

Using (39) and (41), the out-of-plane shear stress components are found to be zero:

\[ \sigma_{zx} = 2G\varepsilon_{zx} = 2G \frac{1}{2} \left( \frac{\partial u_x}{\partial x} + \frac{\partial u_x}{\partial z} \right) = 0, \quad \sigma_{zy} = 2G\varepsilon_{zy} = 2G \frac{1}{2} \left( \frac{\partial u_y}{\partial y} + \frac{\partial u_y}{\partial z} \right) = 0 \]  

(43)

These relations follow from the fact that the \( z \)-component of displacement is zero and the \( x \)- and \( y \)-components of displacement are not functions of \( z \). Then, the in-plane stress components are found directly from (39). Summarizing, the stress components under the conditions of plane strain are:

\[ \sigma_{xx} = (2G + \lambda)\varepsilon_{xx} + \lambda \varepsilon_{yy}, \quad \sigma_{xy} = 2G\varepsilon_{xy}, \quad \sigma_{yy} = (2G + \lambda)\varepsilon_{yy} + \lambda \varepsilon_{xx} \]  

(44)

Given strain components, (44) is used to calculate the stress components.

b) Start with the three-dimensional form of Hooke’s Law for the isotropic elastic material using Young’s modulus, \( E \), and Poisson’s ratio, \( \nu \), as the elastic constants with infinitesimal strain components as dependent variables:

\[ \varepsilon_{ij} = \frac{1 + \nu}{E} \sigma_{ij} - \frac{\nu}{E} \sigma_{kk} \delta_{ij} \]  

(45)

Using the constraints imposed by (38), and the results from part a) of this exercise, expand (45) for the six Cartesian strain components as functions of the stress components.

From (45) the in-plane longitudinal strain in the \( x \)-direction is:

\[ \varepsilon_{xx} = \frac{1 + \nu}{E} \sigma_{xx} - \frac{\nu}{E} \left( \sigma_{xx} + \sigma_{yy} + \sigma_{zz} \right) \]  

(46)

To eliminate \( \sigma_{zz} \) from this equation add the two in-plane normal stresses from (44):

\[ \sigma_{xx} + \sigma_{yy} = 2(2G + \lambda)(\varepsilon_{xx} + \varepsilon_{yy}), \text{ so } \left( \varepsilon_{xx} + \varepsilon_{yy} \right) = \frac{1}{2(2G + \lambda)} \left( \sigma_{xx} + \sigma_{yy} \right) \]  

(47)

Combining (42) and (47), and using (8.28), the out-of-plane normal stress is:

\[ \sigma_{zz} = \frac{\lambda}{2(2G + \lambda)} \left( \sigma_{xx} + \sigma_{yy} \right) \nu \left( \sigma_{xx} + \sigma_{yy} \right) \]  

(48)

Substituting (48) in (46) we have:

\[ \varepsilon_{xx} = \frac{1}{E} \left[ (1 + \nu)\sigma_{xx} - \nu \left[ \sigma_{xx} + \sigma_{yy} + \nu \left( \sigma_{xx} + \sigma_{yy} \right) \right] \right] \]  

(49)

This simplifies to:
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\[ \varepsilon_{xx} = \frac{1}{E} \left[ (1-v^2)\sigma_{xx} - v(1+v)\sigma_{yy} \right] \]  

(50)

The other in-plane longitudinal strain follows similarly, and the in-plane shear strain is found directly from (45). Summarizing, the strain components under the conditions of plane strain are:

\[ \varepsilon_{xx} = \frac{1}{E} \left[ (1-v^2)\sigma_{xx} - v(1+v)\sigma_{yy} \right], \quad \varepsilon_{xy} = \frac{1+v}{E} \sigma_{xy} \]  

(51)

\[ \varepsilon_{yy} = \frac{1}{E} \left[ (1-v^2)\sigma_{yy} - v(1+v)\sigma_{xx} \right], \quad \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0 \]

For a traction boundary value problem (51) would be used to calculate the strain components from the stress components, found by solving the problem.

c) Explain how St. Venant’s six equations of compatibility (7.143) – (7.148) for the infinitesimal strain components reduce to one equation relating the in-plane strain components under conditions of plane strain. Transform this compatibility equation so that it is written in terms of the stress components.

From the definition of plane strain (38) we know that \( u_x \) and \( u_y \) are not functions of \( z \), and that \( u_z \) is identically zero. Referring to the kinematic equations (41) relating infinitesimal strain and displacement components, these restrictions on the displacement components lead to the following restrictions on the strain components:

\[ \varepsilon_{xx} = \varepsilon_{xy}(x, y), \quad \varepsilon_{xy} = \varepsilon_{yy}(x, y), \quad \varepsilon_{yy} = \varepsilon_{yx}(x, y), \quad \varepsilon_{xx} = \varepsilon_{yy} = \varepsilon_{zz} = 0 \]  

(52)

In other words the in-plane strain components are only functions of \( x \) and \( y \), and the out-of-plane strain components are all identically zero. Inspection of (7.143) – (7.148) shows that some of the terms are zero because they are derivatives of the out-of-plane strain components which themselves are zero. Other terms are zero because the are derivatives of the in-plane strain components with respect to \( z \). The only terms that survive are those in the first equation of compatibility:

\[ \frac{\partial^2 \varepsilon_{xx}}{\partial y^2} + \frac{\partial^2 \varepsilon_{xy}}{\partial x^2} + \frac{\partial^2 \varepsilon_{xy}}{\partial x \partial y} = 0 \]  

(53)

Note that we have replaced the material coordinates \( (X, Y, Z) \) used in (7.143) with the spatial coordinates \( (x, y, z) \) in keeping with the consequences of infinitesimal strains.

The strain components given in (51) are substituted into (53) to transform this compatibility equation into one in terms of the stress components. The common factor \( (1+v)/E \) is eliminated to find:

\[ \frac{\partial^2}{\partial y^2} \left[ (1-v)\sigma_{xx} - v\sigma_{yy} \right] + \frac{\partial^2}{\partial x^2} \left[ (1-v)\sigma_{yy} - v\sigma_{xx} \right] - 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0 \]  

(54)

The next step requires one to use the equilibrium equations for this two-dimensional problem which are given in (8.46) and (8.47):

\[ \frac{\partial \sigma_{xx}}{\partial x} + \frac{\partial \sigma_{yx}}{\partial y} = 0, \quad \frac{\partial \sigma_{xy}}{\partial x} + \frac{\partial \sigma_{yy}}{\partial y} = 0 \]  

(55)
Differentiating the first of these with respect to $y$ and the second with respect to $x$ we have:

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0, \quad \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} = 0 \quad (56)$$

Adding these two equations:

$$\frac{\partial^2 \sigma_{xx}}{\partial x^2} + \frac{\partial^2 \sigma_{yy}}{\partial y^2} + 2 \frac{\partial^2 \sigma_{xy}}{\partial x \partial y} = 0 \quad (57)$$

Adding (54) and (57) and eliminating the common factor $(1-\nu)$ we find:

$$\frac{\partial^2}{\partial y^2} \left[ \sigma_{xx} + \sigma_{yy} \right] + \frac{\partial^2}{\partial x^2} \left[ \sigma_{xy} + \sigma_{yx} \right] = \left( \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} \right) \left( \sigma_{xx} + \sigma_{yy} \right) = 0 \quad (58)$$

This is the compatibility equation, written in terms of the stress components, for plane strain with constant body forces.

d) Consider the following Airy stress function:

$$\Phi(x, y) = \frac{1}{6} Cy^3 \quad (59)$$

Derive equations for the three in-plane stress components ignoring body forces. Take the region of interest as the rectangle drawn in Figure 11. Sketch and label the traction boundary conditions acting on this region.

![Figure 11. Region of the $(x, y)$-plane where Airy stress function (59) is used to calculate the stress, strain and displacement components.](image)

The in-plane stress components are found using appropriate derivatives of the stress function as given in (8.35):

$$\sigma_{xx} = \frac{\partial^2 \Phi}{\partial y^2} = \frac{\partial}{\partial y} \left( \frac{1}{2} Cy^3 \right) = Cy, \quad \sigma_{xy} = -\frac{\partial^2 \Phi}{\partial x \partial y} = 0, \quad \sigma_{yy} = \frac{\partial^2 \Phi}{\partial x^2} = 0 \quad (60)$$

Cauchy’s Formula (6.40) in two dimensions reduces to:

$$t_x = \sigma_{xx} n_x + \sigma_{xy} n_y$$

$$t_y = \sigma_{xy} n_x + \sigma_{yy} n_y \quad (61)$$

Taking the stress components from (60) and noting that $n_x = -1$ on $x = 0$, but $n_x = +1$ on $x = L$, the only non-zero tractions for this problem are:

$$t_x = \sigma_{xx} n_x = -Cy \quad (62)$$

$$t_x = \sigma_{xx} n_x = Cy$$
These tractions are plotted schematically in Figure 11 which shows their linear distribution with distance from the center line \((y = 0)\) of the rectangular region and the anti-symmetry across this line.

e) Derive equations for the in-plane strain components associated with the stress distribution given in part d). Use the kinematic equations to derive the displacement components from the strains and cast these equations into dimensionless form. Plot and describe the two normalized displacement component fields and the normalized displacement vector field for the region shown in Figure 11 where \(L = 2H\). Plot the deformed shape of the originally rectangular region. Explain why this solution is called “pure bending”.

Substituting the stress components (60) into Hooke’s Law for this special case of plane strain (51) we have:

\[
\varepsilon_{xx} = \frac{1}{E}\left[(1-v^2)Cy\right], \quad \varepsilon_{yy} = 0, \quad \varepsilon_{xy} = \frac{1}{E}\left[-v(1+v)Cy\right] \tag{63}
\]

The displacement components are found from the kinematic equations by integration:

\[
u_x = \int \varepsilon_{xx} \, dx = \frac{(1-v^2)Cy}{E} \int dx + f_1(y) + C_1 \tag{64}
\]

\[
u_y = \int \varepsilon_{yy} \, dy = -\frac{v(1+v)Cy}{E} \int dy + f_2(x) + C_2 \tag{65}
\]

Because the shear strain is zero using the third of (8.33) with (64) and (65) we have:

\[
\frac{\partial \nu_x}{\partial y} + \frac{\partial \nu_y}{\partial x} = \frac{(1-v^2)Cx}{E} + \frac{df_1(y)}{dy} + \frac{df_2(x)}{dx} = 0 \tag{66}
\]

This condition is possible only if:

\[
\frac{1}{E}\left[(1-v^2)Cx^2\right] + \frac{df_2(x)}{dx} = C_3, \quad \text{and} \quad \frac{df_1(y)}{dy} = -C_3 \tag{67}
\]

Integrating the first of (67) with respect to \(x\), we have:

\[
f_2(x) = -\frac{(1-v^2)Cx^2}{2E} + C_3x + C_4 \tag{68}
\]

Integrating the second of (67) with respect to \(y\), we have:

\[
f_1(y) = -C_3y + C_5 \tag{69}
\]

Substituting (69) into (64) and (68) into (65):

\[
u_x = \frac{(1-v^2)Cxy}{E} - C_3y + C_6 \tag{70}
\]

\[
u_y = -\frac{v(1+v)Cy^2}{2E} - \frac{(1-v^2)Cx^2}{2E} + C_3x + C_7 \tag{71}
\]
The last two terms in each equation for the displacement components account for rigid body rotation and translation respectively. These may be ignored in favor of those terms that account for displacements due to deformation:

\[
\frac{u_E}{(1+v)} = (1-v)x, \quad \frac{u_E}{(1+v)} = \frac{1}{2}(-(1-v)x^2 - vy^2)
\]

(72)

Note that the constant C has units of stress divided by length (MPa/m).

The Matlab m-script `fig_08_sol_12.m` was used to prepare the four parts of Figure 12.

On the first of Figure 12 note that the x-component of displacement is zero along the x-axis and is anti-symmetric about this axis, being positive for \( y > 0 \) and negative for \( y < 0 \). This component increases in magnitude with distance from both coordinate axes. Because this component is zero along the y-axis these displacements are associated with an extension above the middle plane (x-axis) and a contraction below. The middle plane may be referred to as the 'neutral' plane because it does not change length. On the second of Figure 12 note that the y-component of displacement is zero along the y-axis, is negative for all \( x > 0 \), and increases in magnitude with distance from the y-axis. These displacements are associated with a shearing parallel to the y-axis. On the third of Figure 12 the displacement vectors form a pattern that suggests a circulation about the origin in a counterclockwise direction with magnitudes that increase with radial distance. The
deformed shape of the rectangular region is consistent with pure bending in that cross sections remain approximately planar and appear to simply rotate clockwise.
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Figure 12. Pure bending. a) and b) Normalized displacement components. c) Normalized displacement vector field. d) Deformed shape of the rectangular region. Displacements are scaled by a factor $F = 1/3$ to reduce distortion not in keeping with small strain theory.

5) Cylindrical coordinates are the natural system for a number of important problems in structural geology. With zero displacement parallel to the cylindrical $z$-axis these constitute another important set of two-dimensional plane strain problems.

a) The equilibrium equations for plane cylindrical problems are (8.73) and (8.74), and the equations relating the in-plane stress components to the Airy stress functions are (8.75) – (8.77). Show by substitution, while ignoring the body force terms, that these stress components satisfy the equilibrium conditions.

Ignoring body forces, the equilibrium equations for plane cylindrical coordinates are:

$$\frac{\partial \sigma_{rr}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} + \frac{\sigma_{rr} - \sigma_{\theta \theta}}{r} = 0 \quad (73)$$

$$\frac{\partial \sigma_{r\theta}}{\partial r} + \frac{1}{r} \frac{\partial \sigma_{\theta \theta}}{\partial \theta} + \frac{2\sigma_{r\theta}}{r} = 0 \quad (74)$$

The stress components are related to the Airy stress function as:

$$\sigma_{rr} = \frac{1}{r} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}, \quad \sigma_{r\theta} = -\frac{\partial}{\partial \theta} \left( \frac{1}{r} \frac{\partial \Phi}{\partial r} \right), \quad \sigma_{\theta \theta} = \frac{\partial^2 \Phi}{\partial r^2} \quad (75)$$

Note that the shear stress component may be expanded as:

$$\sigma_{r\theta} = -\frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} \quad (76)$$

Substituting for the radial stress component in the first term of (73) we find:

$$\frac{\partial \sigma_{rr}}{\partial r} = \frac{1}{r} \frac{\partial^2 \Phi}{\partial r^2} - \frac{1}{r^2} \frac{\partial \Phi}{\partial r} + \frac{1}{r^2} \frac{\partial^3 \Phi}{\partial r \partial \theta^2} - \frac{2}{r^3} \frac{\partial^2 \Phi}{\partial \theta^2} \quad (77)$$

Substituting for the shear stress component in the second term of (73):

$$\frac{1}{r} \frac{\partial \sigma_{r\theta}}{\partial \theta} = \frac{1}{r} \left\{ \frac{\partial}{\partial \theta} \left[ -\frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r^2} \frac{\partial \Phi}{\partial \theta} \right] \right\} = \frac{1}{r^2} \frac{\partial^3 \Phi}{\partial r \partial \theta^2} + \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} \quad (78)$$
Substituting for the radial and circumferential stress components in the third term of (73):
\[
\frac{\sigma_{rr} - \sigma_{\theta\theta}}{r} = \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial r^2} + \frac{1}{r} \frac{\partial \Phi}{\partial r \partial \theta} - \frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2}
\]
(79)

Adding (77), (78), and (79) all of the terms on the right hand side cancel, thus demonstrating that the first equilibrium equation is identically satisfied using stress components defined as in (75). Substituting for the shear stress component in the first term of (74) we find:
\[
\frac{\partial \sigma_{\theta\theta}}{\partial r} = -\frac{1}{r} \frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{1}{r} \frac{\partial \Phi}{\partial \theta} = -\frac{1}{r} \frac{\partial^3 \Phi}{\partial r^2 \partial \theta} + \frac{2}{r^2} \frac{\partial \Phi}{\partial r \partial \theta} - \frac{2}{r^3} \frac{\partial \Phi}{\partial \theta}
\]
(80)

Substituting for the circumferential stress component in the second term of (74):
\[
\frac{1}{r} \frac{\partial \sigma_{\theta\theta}}{\partial \theta} = \frac{1}{r} \frac{\partial^3 \Phi}{\partial r \partial \theta^2}
\]
(81)

Substituting for the shear stress component in the third term of (74):
\[
\frac{2\sigma_{\theta\theta}}{r} = -\frac{2}{r^2} \frac{\partial^2 \Phi}{\partial r \partial \theta} + \frac{2}{r^3} \frac{\partial \Phi}{\partial \theta}
\]
(82)

Adding (80), (81), and (82) all of the terms on the right hand side cancel, thus demonstrating that the second equilibrium equation is identically satisfied using stress components defined as in (75).

b) Consider the Airy stress function for the stress perturbation due to a cylindrical valley that is 100m deep cut from an elastic half-space (Figure 13) with mass density, \( \rho \), and uniform gravitational acceleration, \( g^* \):
\[
\Phi(r, \theta) = \frac{1}{2} \rho g^* R^2 r \theta \cos \theta
\]
(83)

Derive the equations for the in-plane stress components and write a MATLAB script to plot a contour map of the radial component, \( \sigma_{rr} \). Describe the distribution of the radial stress and explain why this stress component is tensile everywhere.

Figure 13. Schematic illustration of elastic half-space with half-cylindrical cut. Both the half-space surface and the cut are traction free.

The first term in the first equation of (75) evaluates as:
\[
\frac{1}{r} \frac{\partial \Phi}{\partial r} = \frac{1}{2} \rho g^* \left( \frac{R^2}{r} \right) \theta \cos \theta
\]
(84)
The second term evaluates as:
\[
\frac{1}{r^2} \frac{\partial^2 \Phi}{\partial \theta^2} = \frac{1}{2} \rho g \left( \frac{R^2}{r} \right) \frac{\partial}{\partial \theta} \left( -\theta \sin \theta + \cos \theta \right) = \frac{1}{2} \rho g \left( \frac{R^2}{r} \right) \left( -\theta \cos \theta - 2 \sin \theta \right)
\] (85)

Adding (84) and (85) and rearranging we find the radial stress component and the other two stress components are zero by inspection of (75):
\[
\sigma_r = -\rho g \left( r \sin \theta \right) \left( \frac{R}{r} \right)^2, \quad \sigma_{r \theta} = 0, \quad \sigma_{\theta \theta} = 0
\] (86)

Although the radial stress appears to be negative (compressive) note that the coordinate system in Figure 13 is such that \( \pi \leq \theta \leq 2\pi \) for locations within the half-space so \( \sin \theta \leq 0 \) and the stress is positive (tensile).

The Matlab m-script `fig_08_sol_1456.m` plots the contour map of the radial stress component due to the valley (Figure 14).

```matlab
% fig_08_sol_1456.m
% plot stress components near cylindrical valley
clear all, clf
rad = 100; % valley radius, m
rhog = 0.025; % weight per unit volume, MPa/m
x = linspace(0,500,250)+eps; % add eps to avoid singularity at origin
y = linspace(0,-500,250);
[X,Y] = meshgrid(x,y); % define Cartesian grid
[TH,R] = cart2pol(X,Y); % convert to polar coordinates
SRRV = -(rhog*R.*sin(TH)).*((rad./R).^2); % stress due to valley
SRR = (rhog*R.*sin(TH)).*((1-(rad./R).^2)); % add lithostatice stress
STT = rhog*R.*sin(TH); STT(find(R<rad)) = nan;
SRR(find(R<rad)) = nan; SRRV(find(R<rad)) = nan;
contourf(X,Y,SRRV,10), colorbar, axis equal tight;
title('stress srr (valley)'), xlabel('x (m)'), ylabel('y (m)');
figure, contourf(X,Y,SRR,10), colorbar, axis equal tight;
title('stress srr=s1'), xlabel('x (m)'), ylabel('y (m)');
figure, contourf(X,Y,STT,10), colorbar, axis equal tight;
title('stress stt=s2'), xlabel('x (m)'), ylabel('y (m)');

STH = sin(TH); STH2 = STH.^2; CTH = cos(TH); CTH2 = CTH.^2;
SXX = SRR.*CTH2+STT.*STH2; % calculate Cartesian components
SYY = SRR.*STH2+STT.*CTH2;
SXY = (SRR-STT).*CTH.*STH;
G1 = 0.5*atan2(2*SXY,SXX-SYY); % calculate the trajectory angle
G1(find(R<rad)) = nan; U = cos(G1); V = sin(G1);
figure, quiver(X,Y,U,V,.4,'.'); axis equal tight;
title('stress trajectories'), xlabel('x (m)'), ylabel('y (m)');
```

The radial stress perturbation is concentrated at the bottom of the cylindrical cut and decays with radial distance as \((1/r^2)\). It is zero everywhere on the original surface of the half space, but that is not a boundary condition because this normal stress acts tangentially to that surface. The radial stress is tensile everywhere along the cut because it must be equal and opposite to the compressive stress acting on the prospective cut due to the weight of the overlying material.
c) Add the stress state due to weight of the material in absence of the valley to the result from part b) and plot contour maps of the radial and circumferential stress components, $\sigma_{rr}$ and $\sigma_{\theta\theta}$. Indicate where these components match the traction boundary conditions. Explain why the contours of the circumferential component are horizontal lines.

The cylindrical stress components for the half space under gravitational loading only are:

$$\sigma_{rr} = \rho g^* r \sin \theta = \sigma_{\theta\theta}, \quad \sigma_{r\theta} = 0$$  \hspace{1cm} (87)

Adding the perturbing stress due to the cylindrical cut (86) we have:

$$\sigma_{rr} = \rho g^* (r \sin \theta) \left[ 1 - \left( \frac{R}{r} \right)^2 \right], \quad \sigma_{\theta\theta} = \rho g^* (r \sin \theta), \quad \sigma_{r\theta} = 0$$  \hspace{1cm} (88)

The Matlab m-script fig_08_sol_1456.m (see above) plots contour maps of these stress components (Figure 15).

In Figure 15a note that the radial stress is zero everywhere on the cylindrical cut (a boundary condition) and on the surface of the half-space (not a boundary condition). The radial stress is compressive everywhere and of a lesser magnitude than the circumferential stress. The contours of radial stress are depressed under the valley and this depression of the contours decays with depth, illustrating the area of influence of the valley on the lithostatic stress state. Near the bottom of the image the contours are nearly horizontal, just what one would expect for a lithostatic state of stress.

In Figure 15b note that the circumferential stress is non-zero along the cut (not a boundary condition) and is zero along the surface of the half space (a boundary condition). The contours of circumferential stress are horizontal lines indicating that this
stress component is equivalent to the normal stress in absence of the valley. This is the case because the perturbing stress state does not introduce any circumferential stress, only a radial stress.

Figure 15. Normal stress components for the cylindrical valley with lithostatic stress. a) Radial stress. b) Circumferential stress. The valley is 100 m deep and the unit weight of the rock is $\rho g^* = 25$ MPa/km.

d) Consider the total state of stress due to the weight of the half-space and the perturbation of the valley and plot the stress trajectories. What simple relationship do the trajectories have to the cylindrical coordinate system? Why?
The Matlab m-script fig_08_sol_1456.m (see above) plots the trajectories of the principal stresses (Figure 16). The radial stress is the maximum principal stress, $\sigma_1$, and the circumferential stress in the minimum principal stress, $\sigma_2$. Only the maximum principal stress is shown as tic marks in the figure and these are radial everywhere. At any point in the body before the cylindrical valley is cut out, the stress state is isotropic (87). The perturbing solution adds only a radial stress at every point, so that must be one of the principal stresses. Because the perturbing stress is positive (tensile) the radial stress is the maximum principal stress.

![Principal stress trajectories](image)

Figure 16. Maximum principal stress trajectories for the cylindrical valley 100 m deep.

6) Uniaxial compression test results are given for two granites and for a particular limestone measured in two different parts of the same specimen. For both figures the axial stress, $\sigma_a$, is plotted versus the axial extension, $e_a$. Note that both the applied stress and the resulting extension are negative. That is, the stress is compressive and the extension is a shortening.

a) Use the stress-extension graph for Georgia granite to estimate the apparent Young's modulus. Compare your value with the range of values given in Table 8.2 for granites and describe the modulus of Georgia granite relative to those. Would you hear a “ring” or a “thud” if Georgia granite were struck with a geologist’s hammer?
The data for the Georgia granite fall nearly along a straight line. Taking values for the stress and extension at both ends of the data range, we have:

\[ E = \frac{\Delta \sigma}{\Delta \varepsilon} \approx \frac{-125 \text{ MPa}}{-0.002} \approx 60 \text{ GPa} \]  

(89)

The Georgia granite is near the upper end of the range for granites in Table 8.2 (10 – 74 GPa). It is stiffer than the mean value (45 GPa) and softer than the greatest value. Because of this relatively great stiffness you likely would hear a “ring” when a geologist’s hammer struck an outcrop of the Georgia granite.

b) Stress-extension data from a uniaxial compression test on Colorado granite is found in the file colorado.txt with axial stress (MPa) in the first column and axial extension in the second column. Plot these data with stress (ordinate) as a function of extension (abscissa). Use a forward finite difference method to calculate the tangent elastic modulus for each value of the axial extension, omitting that at the origin. Plot the tangent elastic modulus versus axial extension. Describe the apparent non-linear behavior of this rock and provide possible explanations?

The Matlab m-script colorado.m reads the data and plots axial stress versus axial extension (Figure 17a).

% colorado.m  
% Calculate tangent elastic modulus for Colorado Granite  
clear all; clf; reset  
%S,E=textread('colorado.txt','%n%n'); % read data file  
plot(E,S,'bo'), title('Colorado Granite');  
xlabel('Axial Extension'), ylabel('Axial Stress (MPa)');  
E=flipud(E); S=flipud(S); % flip data vectors  
for k = 1:length(S)-1  
\[ E_Y(k) = \frac{E(k+1) - E(k)}{S(k+1) - S(k)} \]  
end  
figure, plot(EY,Y./1000,'ro')  
xlabel('Axial Extension'), ylabel('Tangent Elastic Modulus (GPa)');  
axis([-0.003 0 0 55]), title('Colorado Granite');

Given discrete data approximating a continuous function \( y = f(x) \), the forward difference method calculates an estimate for the slope at the \( i \)th data point as:

\[ \frac{\Delta y}{\Delta x} \approx \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \]  

(90)

Here it is understood that the data are ordered such that \( x_{i+1} > x_i \), so a positive slope corresponds to \( y_{i+1} > y_i \) . Note that this is not the order of the data as given. The Matlab m-script colorado.m flips the data vectors and computes the tangent modulus and plots the forward difference slope versus the axial extension (Figure 17b).

The tangent elastic modulus for the Colorado granite increases almost linearly from 20 GPa to 52 GPa as the specimen contracts from -0.0005 to -0.0016. Then the modulus decreases, again almost linearly, to 35 GPa at a contraction of -0.0030. This range of
elastic moduli is entirely within the range (10 GPa to 74 GPa) for granites quoted in Table 8.2. One explanation for the softer behavior at early stages of the test would be that the platens were not properly seated on the specimen. Later in the test the stiffening could be attributed to closing of cracks oriented more-or-less perpendicular to the specimen axis. Still later in the test the softening could be attributed to the opening of new microcracks, presumably oriented more-of-less parallel to the specimen axis.

Figure 17. Uniaxial compression test data for Colorado granite taken from the file colorado.txt. a) Axial stress versus extension. b) Tangent modulus versus extension.

c) Data in the files limestone.txt and lmst_cracks.txt were taken using samples from the same limestone formation. The axial stress is in the first column and the axial
extension is in the second column of the data files. Construct a plot of axial stress versus axial extension. Describe three differences between mechanical responses of the two limestones. Determine how the tangent modulus differs for each test and plot the tangent modulus versus the axial extension. Suggest what micro-mechanical mechanisms might explain the differences in the tangent moduli.

The Matlab m-script limestone.m reads the data from limestone.txt and makes the stress versus extension plot (Figure 18a). A similar m-script makes the plot for lmst.cracks.txt (Figure 18b).

```matlab
% limestone.m
% Calculate tangent elastic modulus for limestone
clear all, clf reset % clear functions and figures
[S,E]=textread('limestone.txt','%n%n'); % read data file
plot(E(1:7),S(1:7),'bo-',E(7:13),S(7:13),'g*-');
xlabel('Axial Extension'), ylabel('Axial Stress (MPa)');
axis([-0.0025 0 -35 0]), title('Limestone');

EF = flipud(E(1:7)); SF = flipud(S(1:7));
for k = 1:6 % calculate tangent modulus for loading
    EY(k) = EF(k);
    Y(k) = (SF(k+1)-SF(k))/(EF(k+1)-EF(k));
end
for k = 7:12 % calculate tangent modulus for unloading
    EY(k) = E(k);
    Y(k) = (S(k+1)-S(k))/(E(k+1)-E(k));
end
Y = Y/1000; % scale modulus to GPa
figure, plot(EY(1:6),Y(1:6),'bo-',EY(7:12),Y(7:12),'g*-')
xlabel('Axial Extension'), ylabel('Tangent Elastic Modulus (GPa)');
axis([-0.0025 0 0 100]), title('Limestone');
```

![Graph showing stress versus extension for limestone](image.png)
Figure 18. Uniaxial compression test data: blue is loading; green is unloading. a) Limestone taken from the file limestone.txt. b) Limestone with cracks taken from the file lmst_cracks.txt.

Note that Figures 18 a and b are plotted with the same stress and extension scales so one can compare the data for these two samples. For example, at an axial stress of -30 MPa the limestone has contracted by about -0.0005 whereas the limestone with cracks has contracted by about -0.002, a factor of 4 greater. Also, there is a greater difference between the loading (blue) and unloading (green) parts of the tests for the limestone with cracks. Although the zero load data are not included, extrapolating the curves suggests that the limestone with cracks had a greater permanent contraction upon unloading.
The Matlab m-script limestone.m (see above) makes the plot of tangent moduli versus extension (Figure 19a). A similar m-script does the same for lmst_cracks.txt (Figure 19b). Note that the tangent moduli measured immediately after reversing the loading is much greater than any other values for both samples. This suggests a possible error in measurement so those values will be ignored. The tangent moduli for the limestone sample range from 46 GPa to 60 GPa, whereas the tangent moduli for the limestone with cracks range from 10 GPa to 40 GPa. There is no overlap in these ranges and, at a given axial extension, the limestone with cracks is softer by a factor up to nearly 5. One mechanism that could be responsible for the softer behavior is the closure of cracks that are oriented approximately perpendicular to the direction of compression. For both samples the tangent moduli are less near the end of unloading than at the beginning of loading, suggesting that the samples were damaged by the testing.

7) In this exercise the two-dimensional solution for the elastic boundary-value problem of a cylindrical inclusion is used to study possible states of stress both within and near heterogeneities in rock that would serve to concentrate or diminish a remotely applied stress.

a) Investigate the stress state within the inclusion as a function of the shear moduli inside and outside the inclusion. Apply a uniaxial remote stress of unit magnitude in the x-direction. Set Poisson's ratio inside and outside the inclusion to 0.25 and vary the shear moduli to consider the range from an open cavity to a rigid inclusion. Plot your results and describe how the stress state varies with the ratio of shear moduli. What conclusions can you draw from this study about stress concentration and stress diminution within the inclusion?
The Matlab m-script fig_sol_20.m calculates the principal stresses within the inclusion and makes a plot of stress versus the ratio of the shear moduli (Figure 20).

```
% fig_sol_20.m
% Stress inside circular inclusion
clear all, clf reset % clear functions and figures
s1r = 1; s2r = 0; % remote principal stress in x and y
pri = 0.25; prs = 0.25; % Poisson's ratios
ki = 3-4*pri; ks = 3-4*prs;
K = linspace(0,25,101); % ratio of shear moduli
N1 = (K*(ks+2)+ki).*((K*(ks+1)));
N2 = (K*(ks-2)-ki+2).*((K*(ks+1)));
D = eps + 2*(2*K+ki-1).*((K*ks+1));
SXX = (N1./D)*s1r+(N2./D)*s2r; % normal stress in x
SYY = (N2./D)*s1r+(N1./D)*s2r; % normal stress in y
plot(K,SXX,'b-',K,SYY,'r-'), axis([0 25 -0.5 2])
xlabel('Gi/Gs'), ylabel('Sxx and Syy'), title('Inclusion Stress')
```

Figure 20. Plot of stress components $\sigma_{xx}^i$ and $\sigma_{yy}^i$ inside the circular inclusion as a function of the ratio of shear moduli, $G_i/G_s$. Blue curve is $\sigma_{xx}^i = \sigma_i^i$ and red line is $\sigma_{yy}^i = \sigma_2^i$ for uniaxial remote load of unit magnitude in the x-direction.

The two normal stress components in the coordinate directions are principal stresses. For this particular choice of Poisson’s ratios the normal stress in the direction perpendicular to the applied stress is exactly zero for all ratios of shear moduli. In other words it is unchanged from the value in the remote field. For the open cavity, $G_i/G_s = 0$, both stress components inside the inclusion are zero. For the soft inclusion, $G_i/G_s < 0$, the normal stress in the direction of the applied stress is diminished. For a homogeneous material, $G_i/G_s = 1$, both stress components are equal to their respective remote values. For a very stiff inclusion relative to the surroundings, $G_i/G_s >> 1$, the normal stress in the direction of the applied load is concentrated by a factor of 1.5 relative to the remote value.
b) Study the variation of stress within the inclusion for the same conditions as in part a) but set Poisson’s ratio inside and outside the inclusion to 0.5 (incompressible) and then to 0 (perfectly compressible). Plot your results as a function of the shear moduli ratio and describe how the stress state varies for the two different values of Poisson’s ratio. What conclusions can you draw from this study about stress concentration and diminution?

The Matlab m-script fig_sol_20.m (see above) with appropriate changes of Poisson’s ratios calculates the principal stresses within the inclusion and makes the plots of stress versus the ratio of the shear moduli (Figure 21a,b).

![Graph](image-url)
Figure 21. Plot of stress components, $\sigma_{xx}' = \sigma_1'$ and $\sigma_{yy}' = \sigma_2'$, inside the circular inclusion as a function of the ratio of shear moduli, $G_i/G_s$, for incompressible and perfectly compressible materials. Blue curves are $\sigma_{xx}' = \sigma_1'$ and red curves are $\sigma_{yy}' = \sigma_2'$ for uniaxial remote load of unit magnitude in the $x$-direction.

The normal stress in the direction of the applied stress is diminished for the soft inclusion, $G_i/G_s < 0$, and concentrated for the stiff inclusion, $G_i/G_s > 0$, for both extreme values of Poisson’s ratio. However, for the limiting case of the rigid inclusion the stress components approach different values depending upon Poisson’s ratio:

$$ as \ G_i/G_s \rightarrow \infty$$

$$ \sigma_1' \rightarrow \frac{5}{3} \sigma_1' \quad \text{and} \quad \sigma_2' \rightarrow \frac{1}{3} \sigma_1' \quad \text{for} \quad \nu = 0 $$

$$ \sigma_1' \rightarrow \frac{3}{2} \sigma_1' \quad \text{and} \quad \sigma_2' \rightarrow -\frac{1}{2} \sigma_1' \quad \text{for} \quad \nu = \frac{1}{2} $$

The greatest stress concentration (5/3) is for perfectly compressible materials and the greatest stress diminution (-1/2) is for the incompressible materials. In fact, for the incompressible materials, the normal stress acting perpendicular to the applied stress takes on the opposite sign so an applied compression would induce a local tension of one half the magnitude. For the perfectly compressible materials this stress is opposite in sign to the applied stress if the inclusion is softer than the surroundings, but it has the same sign for stiffer inclusions. Clearly the inclusion problem has many interesting results, only a few of which have been explored here!

c) The boundary conditions at the contact of the inclusion with the surroundings specify matching displacements, as though the two materials were tightly bonded together. What does this imply about the tractions acting on the surfaces of the two bodies in contact? What can you deduce about the stress states adjacent to these surfaces? Illustrate your answer with a sketch of the boundary and small volume elements with the appropriate cylindrical components of stress. If there is a discontinuity in any of the components, how does this vary with position on the interface? Illustrate your answers with a plot of the three cylindrical stress components just inside and just outside of the contact as a function of position, $\theta$, using the following parameters:

$$ G_i = 10 \, \text{GPa}, \ G_s = 30 \, \text{GPa}, \ \nu_i = 0.1, \ \nu_s = 0.3, \ \sigma_1' = 1, \ \sigma_2' = 0 $$

Use this plot to demonstrate that your code returns the correct boundary conditions at the interface. Explain why the apparent variation of stress inside the inclusion actually represents a homogeneous state of stress?

At any two adjacent points on opposite sides of the perfectly bonded interface (Figure 22) the traction acting on the inclusion must be equal and oppositely directed to the traction acting on the surroundings. This implies that the following cylindrical stress components on elements in the inclusion and surroundings are equal:

$$ \sigma_{rr}(R^-,\theta) = \sigma_{rr}(R^+\theta) \quad \text{and} \quad \sigma_{r\theta}(R^-,\theta) = \sigma_{r\theta}(R^+\theta) $$

The circumferential normal stresses, $\sigma_{\theta\theta}(R^+,\theta)$ and $\sigma_{\theta\theta}(R^+\theta)$ are not given by the boundary conditions and may be discontinuous across the interface. Note that
conservation of angular momentum and the second of (93) dictates that
\( \sigma_{rr} (R^-, \theta) = \sigma_{rr} (R^+, \theta) \). Thus of the three independent stress components two are
constrained by the boundary conditions at the interface.

Figure 22. Illustration of the elastic inclusion with cylindrical stress components.

The Matlab m-script fig_sol_23.m calculates the stresses within the inclusion and just
outside in the surroundings and makes plots of the stress components versus position
along the interface (Figure 23a,b).

```matlab
% fig_08_sol_23
% Boundary conditions for the circular inclusion
clear all, clf reset % clear functions and figures
ri = 1; s1 = 1; s2 = 0;
pri = 0.1; prs = 0.3; ki = 3-4*pri; ks = 3-4*prs;
mui = 10000; mus = 30000; k = mui/mus;
num1 = (k*(ks+2)+ki)*k*(ks+1); num2 = (k*(ks-2)-ki+2)*k*(ks+1);
den = 2*(2*k+ki-1)*(k*ks+1);
SXX = (num1/den)*s1+(num2/den)*s2;
SYY = (num2/den)*s1+(num1/den)*s2;
TH = 0:pi/30:2*pi
THD = TH*180/pi;
ST = sin(TH); S2T = sin(2*TH); ST2 = ST.^2;
CT = cos(TH); C2T = cos(2*TH); CT2 = CT.^2;
SRRI = SXX.*CT2+SYY.*ST2;
STTI = SXX.*ST2+SYY.*CT2;
SRTI = -(SXX-SYY).*CT.*ST;

```

```matlab
a = 2*(1-k)/(k*ks+1); b = (ki-1-k*(ks-1))/(2*k+ki-1);
c = (k-1)/(k*ks+1);
R = 1; R2 = (ri./R).^2; R4 = R2.^2;
SRR = (0.5*(s1+s2)*((1-2*a*R2-3*c*R4).*C2T));
STT = (0.5*(s1+s2)*((1+b*R2)-(1-3*c*R4).*C2T));
SRT = (-0.5*(s1-s2)*((1+a*R2+3*c*R4).*S2T));
plot(THD,SRR,'bx',THD,STT,'gx',THD,SRT,'rx');
xlabel('theta (degrees)'); ylabel('stress');axis([0 360 -1 2]);
```
Figure 23. Stress variation with angle theta around the cylindrical inclusion both inside (curves) and just outside (crosses) the interface.

Note that the boundary conditions (93) are precisely met by the code. Also note that the circumferential stresses, $\sigma_{\theta\theta}(R^-,\theta)$ and $\sigma_{\theta\theta}(R^+,\theta)$, only match at four locations around the inclusion. The circumferential stress just outside the inclusion has the same sinusoidal distribution as that inside the inclusion but the amplitudes differ. For this case of a softer inclusion the remote stress is concentrated outside the inclusion and obtains a maximum value of $1.92\sigma_i^T$ at $\theta = 90^\circ, 270^\circ$. The greatest diminution also is just outside the inclusion where the stress obtains a minimum value of $-0.33\sigma_i^T$ at $\theta = 0^\circ, 180^\circ$. Inside the inclusion the cylindrical stress components vary in a sinusoidal manner with angle $\theta$. This is the expected variation with orientation of the volume element in a body subject to a homogeneous stress state.

d) Investigate the spatial variation of the stress components inside and just outside the inclusion, $r = R^+$, given the parameters in (92) except vary the inclusion shear modulus to consider three cases: an open cavity; a homogeneous body; and a much stiffer inclusion. Keep track of the position, orientation, and magnitude of the greatest tensile stress and use this to describe where and with what orientation opening cracks would be predicted to form if this tension equals the tensile strength. Now change the applied stress to a unit compression acting in the $x$-direction and address the same questions about opening cracks.

The following results are for a unit remotely applied tension.
For the open cavity the greatest tensile stress is found along the edge of the cavity where
\( \sigma_{\theta\theta} \left( r = R^+ \right) = 3\sigma'_i \). This would result in opening cracks forming at the
dge of the cavity along the y-axis and oriented parallel to the y-axis.

For homogeneous material properties the greatest tensile stress is found everywhere and
is the same magnitude as the applied stress: \( \sigma_{xx} \left( r, \theta \right) = \sigma'_i \). This would result in opening
-cracks throughout the body oriented parallel to the y-axis.

For the very stiff inclusion the greatest tensile stress is found inside the inclusion where
\( \sigma_{xx} \left( r \leq R^+, \theta \right) = 1.4778 \sigma'_i \). Because of the bonded interface this same stress is found just
outside the inclusion where \( \sigma_{rr} \left( r = 0^+, \theta = 0^\circ, 180^\circ \right) = 1.478 \sigma'_i \). This would result in
opening cracks forming everywhere inside the inclusion and just outside the inclusion
along the x-axis and in an orientation parallel to the y-axis.

The following results are for the unit remotely applied compression.

For the open cavity the greatest tensile stress is found along the edge of the cavity where
\( \sigma_{\theta\theta} \left( r = R^+, \theta = 0^\circ, 180^\circ \right) = -10\sigma'_i \). This would result in opening cracks forming at the
dge of the cavity along the x-axis and oriented parallel to the x-axis.

For the homogeneous body the stress is compressive everywhere so this would not result
in opening cracks forming.

For the very stiff inclusion the greatest tensile stress is found inside the inclusion where
\( \sigma_{yy} \left( r \leq R^+, \theta \right) = -0.0778 \sigma'_i \). Because of the bonded interface this same stress is found just
outside the inclusion where \( \sigma_{rr} \left( r = R^+, \theta = 90^\circ, 270^\circ \right) = -0.0778 \sigma'_i \). This would result in
opening cracks forming everywhere inside the inclusion and just outside the inclusion
along the y-axis and in an orientation parallel to the x-axis. However note that the
magnitude of the tensile stress is only about 8% of the applied compression.

e) Investigate the radius of influence of the inclusion on the stress field in the
surrounding material. Because the remotely applied stresses are referred to the
Cartesian coordinate axes use the Cartesian stress components. Consider the
spatial variation of stress components along radial lines extending from the edge
of the inclusion, \( r/R = 1 \), to a distance of six times the inclusion radius. Begin by
considering the case of an open cavity under uniaxial stress of unit magnitude in
the x-direction and use 10% of this stress as the threshold for identifying a
significant perturbation. Determine whether the radius of influence changes for
different stress components and for different orientations of the radial line.
Determine whether the radius of influence is significantly different for the very
stiff inclusion relative to the surroundings.
The Matlab m-script fig_sol_24.m calculates the Cartesian stresses outside the inclusion along radial lines and makes plots of the stress components versus position (Figure 24). The following parameters were chosen for the construction of this figure:

\[ G_f = 0 \text{ GPa}, \quad G_s = 30 \text{ GPa}, \quad \nu_f = 0.1, \quad \nu_s = 0.3, \quad \sigma_1' = 1, \quad \sigma_2' = 0 \]  

(94)

Figure 24a. Spatial variation of Cartesian stress components with radial distance from the cylindrical cavity along the y-axis (\(\theta = 90^\circ\)).

From Figure 24 note that the x-component of normal stress decreases monotonically from \(\sigma_{xx} = 3\sigma_1'\) at the edge of the cavity toward \(\sigma_{xx} = 1\sigma_1'\) far from this edge. The y-component of normal stress increases from zero at the edge to a maximum value and then
decreases toward zero far from the edge. For $r/R > 2.65$ the $x$-component of normal stress is within 10% of the applied stress, whereas the $y$-component falls to 10% of the applied stress for $r/R > 3.70$. In other words the radius of influence in terms of these stress components for this inclusion is about $r = 3.7R$.

On the other hand, changing the orientation of the radial line and using the m-script `fig_sol_24.m` we find the $x$-component of normal stress along the $x$-axis ($\theta = 0^\circ$) increases to within 10% of the remote applied stress for $r/R > 4.90$. The $x$-component of normal stress along the radial line ($\theta = 45^\circ$) decreases to within 10% of the remote applied stress for $r/R > 2.85$. Clearly the perturbation of the stress field by the inclusion depends upon the particular component and the orientation of the radial line. Based upon this limited analysis we would conclude that the radius of influence for the open cylindrical cavity is about $r = 5R$.

![Figure 24b. Spatial variation of Cartesian stress components with radial distance from the rigid cylindrical inclusion along the y-axis ($\theta = 90^\circ$).](image)

For the rigid inclusion along the $y$-axis ($\theta = 90^\circ$) we find the $x$-component of normal stress increases to within 10% of the applied stress for $r/R > 2.00$, whereas the $y$-component increases to 10% of the applied stress for $r/R > 2.80$. The $x$-component of normal stress along the $x$-axis ($\theta = 0^\circ$) decreases to within 10% of the remote applied stress for $r/R > 3.50$. The $x$-component of normal stress along the radial line ($\theta = 45^\circ$) decreases to within 10% of the remote applied stress for $r/R > 1.30$. Based upon this limited analysis we would conclude that the radius of influence for the rigid cylindrical inclusion is about $r = 3.5R$. All of these values are less than those for the open cavity, suggesting that stiffer inclusions have smaller radii of influence than softer inclusions.

8) Use the two-dimensional solution for the elastic boundary-value problem of a cylindrical hole in an orthotropic material to study possible states of stress near holes in
anisotropic rock that would serve to concentrate stress. For plane strain conditions there are two orthogonal axes of elastic symmetry in the \((x, y)\)-plane. \(E_1\) and \(\nu_{12}\) are Young’s modulus and Poisson’s ratio in the \(x\)-coordinate direction and \(E_2\) and \(\nu_{21}\) are the respective moduli in the \(y\)-coordinate direction. The self-consistent shear modulus in the \((x, y)\)-plane is \(G\).

\[ G = 6.0 \text{ GPa, } \nu_{12} = 0.2 \]  
(95)

Calculate the value of the second Poisson’s ratio, \(\nu_{21}\), and write down all five elastic moduli.

For the orthotropic material Young’s modulus and Poisson’s ratios are related as (8.133):

\[
\frac{\nu_{12}}{E_1} = \frac{\nu_{21}}{E_2}
\]  
(96)

Solving for the unknown Poisson’s ratio we have:

\[
\nu_{21} = E_2 \left( \frac{\nu_{12}}{E_1} \right) = 0.12
\]  
(97)

Therefore, the five moduli for the oil shale are:

\[
E_1 = 21.4 \text{ GPa, } E_2 = 12.4 \text{ GPa, } G = 6.0 \text{ GPa, } \nu_{12} = 0.20, \nu_{21} = 0.12
\]  
(98)

b) Write down the linear strain-stress equations using compliances and using the more familiar laboratory constants (Young’s modulus and Poisson’s ratio) for the orthotropic material. Use these equations to derive equations for the constants, \(C_1\) and \(C_2\), employed in solutions to the orthotropic elastic boundary value problem:

\[
C_1 = \frac{s_{11}}{s_{22}}, \quad C_2 = \frac{(s_{66} + 2s_{12})}{s_{22}}
\]  
(99)

The strain-stress equations using compliances are (8.131):

\[
\varepsilon_{xx} = s_{11} \sigma_{xx} + s_{12} \sigma_{yy}, \quad \varepsilon_{yy} = s_{12} \sigma_{xx} + s_{22} \sigma_{yy}, \quad \varepsilon_{xy} = s_{66} \sigma_{xy}
\]  
(100)

The strain-stress equations using laboratory constants are (8.132):

\[
\varepsilon_{xx} = \frac{1}{E_1} \sigma_{xx} - \frac{\nu_{21}}{E_2} \sigma_{yy}, \quad \varepsilon_{yy} = -\frac{\nu_{12}}{E_1} \sigma_{xx} + \frac{1}{E_2} \sigma_{yy}, \quad \varepsilon_{xy} = \frac{1}{G} \sigma_{xy}
\]  
(101)

By inspection of (100) and (101) we may rewrite (99) as:

\[
C_1 = \frac{E_2}{E_1}, \quad C_2 = \frac{(s_{66} + 2s_{12})}{s_{22}} = \frac{(1/G) - 2(\nu_{12}/E_2)}{(1/E_2)} = \frac{E_2}{G} - 2\nu_{12}
\]  
(102)

c) The elastic moduli for the oil shale from part a) must be related to elastic compliances that are real numbers. Test these values to determine if this condition holds starting with the following equations for the constants \(\alpha_1\) and \(\alpha_2\) which
appear in the governing compatibility equation for orthotropic elastic boundary value problems:

\[ C_1 = \alpha_1 \alpha_2, \quad C_2 = \alpha_1 + \alpha_2 \]  

(103)

Place an upper bound on the shear modulus assuming the measured values of the two Young’s moduli and the given value for the Poisson’s ratio are correct.

Multiplying the second of (103) by the constants \( \alpha_1 \) and then subtracting this from the first of (103) we have:

\[ \alpha_1 C_2 = \alpha_1^2 + \alpha_2 \alpha_1 \]  

so \( \alpha_1^2 - C_2 \alpha_1 + C_1 = 0 \)  

(104)

Solving (104) using the standard method for a quadratic equation:

\[ \alpha_1 = \frac{1}{2} C_2 \pm \frac{1}{2} \sqrt{C_2^2 - 4 C_1}, \text{ where } C_2^2 - 4 C_1 \geq 0 \]  

(105)

The inequality relation for the two constants is necessary for the solution of the quadratic to be real numbers. If the constants \( C_1 \) and \( C_2 \) are not real, then by (102) the elastic moduli are not real numbers. Substituting from (102) this condition may be rewritten:

\[ \left( \frac{E_2}{G} - 2\nu_{12} \right)^2 - 4 \left( \frac{E_2}{E_1} \right) \geq 0 \]  

(106)

A plot of the left hand side versus \( G \) is given in Figure 25 where we see that the upper bound on \( G \) is approximately 7GPa for the choices of other moduli used for this oil shale.

Figure 25. Plot of condition number (left hand side of (106)) versus \( G \). Real values for the elastic moduli require that this number be positive.

The Matlab m-script `fig_08_sol_25.m` calculates the condition number and makes a plot of the condition number versus \( G \) (Figure 25).

% fig_08_sol_25.m
clear all, clf reset; % clear memory and figures
G = 0.1:0.1:10; % range of values for G (GPa)
e1 = 21.4; e2 = 12.4; % Young's moduli (GPa)
nu12 = 0.2; nu21 = nu12*e2/e1; % Poisson's ratios
COND = ((e2/G)-2*nu21).^2 - 4*(e2/e1); % cond must be zero or positive
plot(G,COND,'b-'); axis([0 10 -10 20]); hold on
plot([0 10],[0 0],'r-'); title('test for self-consistent $G$');
xlabel('G'); ylabel('Condition Number');

d) Use the elastic moduli found in part a) and calculate the circumferential stress around the circular hole in the orthotropic material for a unit remote normal stress and plot this distribution. Describe the concentration and diminution of stress around the hole. Compare your result to that for an isotropic rock using the Kirsh solution and plot this distribution on the same graph. Note that the Kirsh solution is independent of the elastic moduli. Evaluate the errors introduced in calculations of the stress state if you were to assume the oil shale is isotropic.

The normalized circumferential stress on the edge of the hole is found using (8.146):

$$
\frac{\sigma_{\theta\theta}}{\sigma_i^*} = \frac{(1+\gamma_1)(1+\gamma_2)(1+\gamma_1+\gamma_2-\gamma_1\gamma_2-2\cos 2\theta)}{(1+\gamma_1^2-2\gamma_1\cos 2\theta)(1+\gamma_2^2-2\gamma_2\cos 2\theta)}
$$

(107)

The constants $\gamma_1$ and $\gamma_2$ are related to those defined above as follows. From (102):

$$
C_1 = \frac{E_2}{E_1} = 0.5794, \quad C_2 = \frac{E_2}{G} - 2\nu_{12} = 1.8349
$$

(108)

Based on these values the condition number is 1.0491 so all moduli are real numbers. From (105) and the first of (103):

$$
\alpha_1 = \frac{1}{2} C_2 \pm \frac{1}{2} \sqrt{C_2^2 - 4C_1} = 1.4296, \quad \alpha_2 = C_1/\alpha_1 = 0.4053
$$

(109)

Then, from (8.145):

$$
\gamma_1 = \frac{\sqrt{\alpha_1} - 1}{\sqrt{\alpha_1} + 1} = 0.0891, \quad \gamma_2 = \frac{\sqrt{\alpha_2} - 1}{\sqrt{\alpha_2} + 1} = -0.2220
$$

(110)

From (6.110) the Kirsh solution is taken with $S_H = -\sigma_i^*, S_p = 0, P = 0, r = R$ so we have:

$$
\frac{\sigma_{\theta\theta}}{\sigma_i^*} = 1 - 2\cos 2\theta
$$

(111)

The Matlab m-script fig_08_sol_26.m calculates the circumferential stress at the edge of the circular hole for the isotropic and orthotropic materials (Figure 26).
cond = c2^2 - 4*c1 % NOTE: cond must be zero or positive
a1 = 0.5*c2 + 0.5*sqrt(cond); a2 = c1/a1;
g1 = (a1^0.5 -1)/(a1^0.5 +1); g2 = (a2^0.5 -1)/(a2^0.5 +1);
NUM = (1+g1)*(1+g2)*((1+g1+g2-g1*g2-2*C2T);
DEN = (1+g1^2-2*g1*C2T).*((1+g2^2-2*g2*C2T);
STTA = s1*NUM./DEN; % anisotropic case
plot(THD,STTI,‘g-‘,THD,STTA,’r-‘);
xlabel(‘theta (degrees)‘); ylabel(‘stress/remote compression‘);
legend(‘STTI’,’STTA‘); axis ([0 180 -2 4]);

Figure 26. Normalized circumferential stress at edge of circular hole in isotropic (green curve) and orthotropic (red curve) materials. Remote stress is unit tension.

Note that the stress concentration at $\theta = 90^\circ$ is 3.4071 for the orthotropic material versus 3 for the isotropic material. The error is 14%. The stress diminution at $\theta = 0^\circ$ is -0.7612 for the orthotropic material versus -1 for the isotropic material. The error is 24%.